计算固体力学

(Computational Solid Mechanics)

王永亮

Computational Rock Mechanics Research Group
School of Mechanical and Civil Engineering
China University of Mining and Technology (BJ)

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课程内容

1. **计算固体力学绪论**
2. **微分方程的等效积分弱形式**
3. **弹性力学问题的有限元求解格式**

1. Introduction of Computational Mechanics

Computational mechanics is the discipline concerned with the use of computational methods to study phenomena governed by the principles of mechanics. Before the emergence of computational science (also called scientific computing) as a **"third way" besides theoretical and experimental sciences**, computational mechanics was widely considered to be a sub-discipline of applied mechanics. It is now considered to be a sub-discipline within computational science.

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计算力学 锁定

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科普中国 CHINA SCIENCE COMMUNICATION

Baidu百科

计算力学（computational mechanics）是根据力学中的理论，利用现代电子计算机和各种数值方法，解决力学中的实际问题的一门新兴学科。它横贯力学的各个分支，不断扩大各个领域中力学的研究和应用范围，同时也在逐渐发展自己的理论和方法。计算力学的应用范围已扩大到固体力学、岩土力学、水力学、流体力学、生物力学等领域。计算力学主要进行数值方法的研究，如对有限差分方法、有限元法作进一步深入研究，对一些新的方法及基础理论问题进行探索等等。计算力学横贯各个力学分支，为它们服务，促进它们的发展，同时也受它们的影响。

1. Introduction of Computational Mechanics

- **Numerical methods in computational mechanics**

作为力学分支的计算力学，发展了有限元(**finite element method, FEM**)、离散元(**discrete element method, DEM**)、有限差分法(**finite difference method, FDM**)、无网格法(**mesh-less method, MLM**)、扩展有限元法(**extended finite element method, XFEM**)、边界元(**boundary element method, BEM**)、半解析方法(**semi-analytic methods**）等理论和方法，为虚拟仿真提供了工具。

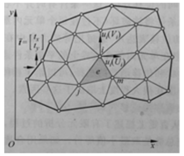
计算固体力学(**Computational Solid Mechanics)**是计算力学下的固体力学研究分支。

1 Finite Element Method

- **Continuous problems to discrete problems**

(a) The *continuum* is divided into a finite number of parts (elements), the behavior of which is specified by a finite number of parameters, and
(b) the solution of the complete system as an assembly of its elements follows precisely the same rules as those applicable to *standard discrete problems*.

- **The main procedure of finite element method**



- 建立模型: **Pre-process(前处理)**
- 控制方程: **Continuum problems** **(Partial Differential Equations, PDEs)**
- 离散方程: **Discrete problems** **(Algebraic Equations, AEs)**
- 求解方法: **Gaussian elimination methods etc.**
- 展示结果: **Post-process(后处理)**

From Continuum to Discrete

课程内容

1. 计算固体力学绪论
2. 微分方程的等效积分弱形式
3. 弹性力学问题的有限元求解格式

2 Computational Solid Mechanics

2.1 Elasticity equations

2.2 Weak Forms and FE Approximation:

1-D Problems

2.1 Elasticity equations

- **Basic equations**

- The basic equations for the theory of elasticity are described in terms of **displacements, strains, stresses, boundary conditions, and constitutive relations** that relate the behavior between strain and stress.

- We start by specifying each equation set for a general three-dimensional problem in Cartesian coordinates. However, we will also consider some two-dimensional forms. The two-dimensional problems we consider are of three types: ***plane stress***, ***plane strain*** and ***axisymmetric cases.***

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2.1 Elasticity equations

■ Two-dimensional problems

a) The ***plane stress*** case. In this problem the only nonzero stresses are those in the plane of the problem and normal to the lamina we have no stresses as shown in Fig. 2.1a.
b) The ***plane strain*** case. Here all straining normal to the plane considered is prevented. Such a situation may arise in the long prism shown in Fig. 2.1b in which loading does not vary in the direction normal to the plane.
c) The third and final case of two-dimensional analysis is that in which the situation is ***axisymmetric***. Here the plane considered is one at constant θ in a cylindrical coordinate system r–z–θ (Fig. 2.1c) and all components of displacement, stress, and strain are assumed dependent on r and z only.

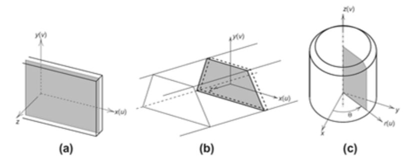

FIGURE 2.1
Two-dimensional analysis types for (a) plane stress, (b) plane strain, and (c) axisymmetry.

2.1 Elasticity equations

2.1.1 Displacement function

- **3D problem**

$$\mathbf{u}(\mathbf{x},\,t)=\begin{Bmatrix} u(x,\,y,\,z,\,t) \\ v(x,\,y,\,z,\,t) \\ w(x,\,y,\,z,\,t) \end{Bmatrix} \qquad \mathbf{x}=\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \qquad t\text{—time}$$

- **2D problem**

✓ ***plane stress and plane strain*** cases

$$\mathbf{u}(\mathbf{x},\,t)=\begin{Bmatrix} u(x,\,y,\,t) \\ v(x,\,y,\,t) \end{Bmatrix}$$

✓ ***axisymmetric*** case

$$\mathbf{u}(\mathbf{x},\,t)=\begin{Bmatrix} u(r,\,z,\,t) \\ v(r,\,z,\,t) \end{Bmatrix} \qquad \mathbf{x}=\begin{Bmatrix} r \\ z \end{Bmatrix}$$

2.1 Elasticity equations

2.1.2 Strain matrix

- In a three-dimensional problem there are six independent components of strain which we order and denote in matrix form by

$$\boldsymbol{\varepsilon}=\begin{bmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_z & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix}^{\mathrm{T}}$$

  This form is known in the mechanics literature as Voigt notation [8]. It is a way of writing a symmetric second order tensor in terms of a reduced set of components. The strain is a symmetric form where $\gamma_{xy}=\gamma_{yx}$, $\gamma_{yz}=\gamma_{zy}$, and $\gamma_{zx}=\gamma_{xz}$; thus, Voigt notation reduces nine components to six.
- For the two-dimensional problems considered in this volume the last two components are always zero. Thus, only four components of ε need be considered.

2.1 Elasticity equations

■ Strain-displacement matrix

- For convenience in considering all three classes of two-dimensional problems in a unified manner, we include four components of strain in $\boldsymbol{\varepsilon}$ and write them as

$$\boldsymbol{\varepsilon}=\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \end{bmatrix}=\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix}+\begin{Bmatrix} 0 \\ 0 \\ \varepsilon_z \\ 0 \end{Bmatrix}=\boldsymbol{\mathcal{S}}_p\,\mathbf{u}+\boldsymbol{\varepsilon}_z$$

for plane problems (where $\boldsymbol{\varepsilon}_z$ is zero for plane strain but not for plane stress) and

$$\boldsymbol{\varepsilon}=\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix}=\begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix}=\boldsymbol{\mathcal{S}}_a\,\mathbf{u}$$

2.1 Elasticity equations

■ Strain-displacement matrix

- The strains for a problem undergoing small deformations are computed from the displacements and may be expressed in matrix form as

$$\boldsymbol{\varepsilon} = \boldsymbol{\mathcal{S}}\,\mathbf{u}$$

where $\boldsymbol{S}$ is a matrix of differential operators and $\mathbf{u}$ is the displacement field. For the three-dimensional problem the strain-displacement relations are given by

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

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2.1 Elasticity equations

2.1.3 Stress matrix

- The components σ_x, σ_y, σ_z are called normal stresses and τ_{xy}, τ_{yx}, τ_{yz}, τ_{zy}, τ_{xz}, τ_{zx} are called shearing stresses

$$\tau_{xy} = \tau_{yx},\ \tau_{yz} = \tau_{zy} \text{ and } \tau_{zx} = \tau_{xz}$$

Thus, similar to strain, the stresses may be written in terms of six components that are ordered and denoted in matrix form by

$$\boldsymbol{\sigma} = [\,\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx}\,]^{\mathrm{T}}$$

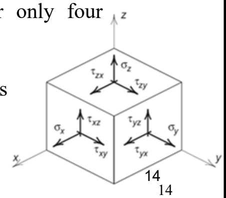
- For two-dimensional plane problems we consider only four components of stress and use

$$\boldsymbol{\sigma} = [\,\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy}\,]^{\mathrm{T}}$$

In axisymmetry we define the components of stress as

$$\boldsymbol{\sigma} = [\,\sigma_r \quad \sigma_z \quad \sigma_\theta \quad \tau_{rz}\,]^{\mathrm{T}}$$

In plane stress problems we know a priori that σ_z= 0.



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2.1 Elasticity equations

2.1.4 Equilibrium equations

- The linear momentum or equilibrium equations for the three-dimensional behavior of a solid may be written in Cartesian coordinates as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + b_x = \rho \frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + b_y = \rho \frac{\partial^2 v}{\partial t^2}$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z = \rho \frac{\partial^2 w}{\partial t^2}$$

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2.1 Elasticity equations

2.1.4 Equilibrium equations

- The equilibrium equations in Cartesian coordinates may be written in a matrix Voigt form as

$$\boldsymbol{\mathcal{S}}^{\mathrm{T}}\boldsymbol{\sigma} + \mathbf{b} = \rho\,\ddot{\mathbf{u}}$$

$\boldsymbol{S}$ is the same differential operator, $\mathbf{b}$ is the vector of body forces given as

$$\mathbf{b} = [\,b_x \quad b_y \quad b_z\,]^{\mathrm{T}}$$

ρ is the mass density per unit volume and $\ddot{\mathbf{u}} = \partial^2\mathbf{u}/\partial t^2$ is the acceleration vector.

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2.1 Elasticity equations

2.1.4 Equilibrium equations

- The linear momentum or equilibrium equations for the two-dimensional plane problems behavior of a solid may be written in Cartesian coordinates as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + b_x = \rho \frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = \rho \frac{\partial^2 v}{\partial t^2}$$

and in matrix form

$$\mathcal{S}_p^{\mathrm{T}} \boldsymbol{\sigma} + \mathbf{b} = \rho\, \ddot{\mathbf{u}}$$

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2.1 Elasticity equations

2.1.4 Equilibrium equations

- The linear momentum or equilibrium equations for the two-dimensional axisymmetric problems behavior of a solid may be written in Cartesian coordinates as

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \tau_{zr}}{\partial z} + b_r = \rho \frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + b_z = \rho \frac{\partial^2 v}{\partial t^2}$$

and the differential operator on equilibrium in matrix form

$$\bar{\mathcal{S}}_a^{\mathrm{T}} = \begin{bmatrix} \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) & 0 & -\frac{1}{r} & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} \end{bmatrix}$$

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2.1 Elasticity equations

2.1.5 Boundary conditions(BCs)

- Displacement boundary conditions are specified at each point of the boundary Γ_u as

$$\mathbf{u} = \bar{\mathbf{u}}(\mathbf{x},\ t)$$

where $\bar{\mathbf{u}}$ are known values and *x* are points on the boundary

- Traction boundary conditions are specified for each point of the boundary **t** and are given in terms of stresses by

$$\mathbf{t} = \mathbf{G}^{\mathrm{T}} \boldsymbol{\sigma} = \bar{\mathbf{t}}(\mathbf{x},\ t)$$

in which for three-dimensional problems $\mathbf{G}^{\mathrm{T}}$ is the matrix, and in two-dimensional plane problems $\mathbf{G}^{\mathrm{T}}$ reduces to $\mathbf{G}_p^{\mathrm{T}}$

$$\mathbf{G}^{\mathrm{T}} = \begin{bmatrix} n_x & 0 & 0 & n_y & 0 & n_z \\ 0 & n_y & 0 & n_x & n_z & 0 \\ 0 & 0 & n_z & 0 & n_y & n_x \end{bmatrix} \qquad \mathbf{G}_p^{\mathrm{T}} = \begin{bmatrix} n_x & 0 & 0 & n_y \\ 0 & n_y & 0 & n_x \end{bmatrix}$$

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2.1 Elasticity equations

2.1.6 Stress and strain relations: Elasticity matrix

- Using this hypothesis the stress-strain equations for a linearly elastic material may be expressed by

$$\boldsymbol{\sigma} = \mathbf{D}\, \boldsymbol{\varepsilon}$$

or by

$$\boldsymbol{\varepsilon} = \mathbf{D}^{-1} \boldsymbol{\sigma}$$

The **D** matrix is known as the elasticity matrix of moduli and the $\mathbf{D}^{-1}$ matrix as the elasticity matrix of compliances.

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2.1 Elasticity equations

■ Isotropic materials

- In cartesian coordinates system

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix}$$

Inverting to obtain the appropriate elasticity matrix of moduli yields the result

$$\mathbf{D} = \frac{E}{d}\begin{bmatrix}(1-\nu) & \nu & \nu & 0 & 0 & 0\\ \nu & (1-\nu) & \nu & 0 & 0 & 0\\ \nu & \nu & (1-\nu) & 0 & 0 & 0\\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0\\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0\\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2\end{bmatrix}$$

$d = (1+\nu)(1-2\nu)$

$-1 < \nu < \frac{1}{2}$

2.1 Elasticity equations

■ Isotropic materials

- Two-dimensional problems in cartesian coordinates system

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\end{Bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu & 0\\ -\nu & 1 & -\nu & 0\\ -\nu & -\nu & 1 & 0\\ 0 & 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\end{Bmatrix}$$

✓ The plane stress case

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\end{Bmatrix} = \frac{E}{(1-\nu^2)}\begin{bmatrix}1 & \nu & 0 & 0\\ \nu & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & (1-\nu)/2\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\end{Bmatrix}$$

✓ The plane strain case

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\end{Bmatrix} = \frac{E}{d}\begin{bmatrix}(1-\nu) & \nu & \nu & 0\\ \nu & (1-\nu) & \nu & 0\\ \nu & \nu & (1-\nu) & 0\\ 0 & 0 & 0 & (1-2\nu)/2\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\end{Bmatrix}$$

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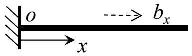
2.1 Elasticity equations

■ One-dimensional form of elasticity



- Equilibrium equations: $\dfrac{\partial \sigma_x}{\partial x} + b_x = \rho\,\dfrac{\partial^2 u}{\partial t^2}$ *Strong form*
- Constitutive equation: $\sigma_x = E\,\varepsilon_x$
-
- Strain-displacement equation: $\varepsilon_x = \dfrac{\partial u}{\partial x}$
- Boundary conditions

n_x is the unit outward normal

$$u = \bar{u} \quad \text{or} \quad t_x = \bar{t}_x = n_x \sigma_x \quad \text{on } x = a, b$$

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The End