计算固体力学

(Computational Solid Mechanics)

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课程内容

1. 计算固体力学绪论
2. 微分方程的等效积分弱形式
3. 弹性力学问题的有限元求解格式

2 Computational Solid Mechanics

2.1 Elasticity equations

2.2 Weak Forms and FE Approximation: 1-D Problems

2.2 Weak form of equivalent integration for differential equations

- Strong Form: governing equations (Partial Differential Equations, PDEs)
- Weak Form:
  (1) Multiply each equation by an appropriate arbitrary function.
  (2) Integrate this product over the space domain of the problem.
  (3) Use integration by parts to reduce the order of derivatives to a minimum.
  (4) Introduce boundary conditions if possible.

An arbitrary function is one that can take any value we can imagine. It can be a polynomial(多项式), a trigonometric function (三角函数), a Dirac delta function, or any other function.

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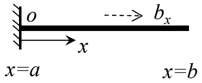
2.2 Weak form of one-dimensional elasticity problems

■ **One-dimensional elastic cantilever beam**



- Displacemet: u strain: ε_x stress: σ_x
- Geometric equation: $\varepsilon_x = \frac{\partial u}{\partial x}$
- Constitutive equation: $\sigma_x = E\varepsilon_x$
- Equilibrium equation: $\frac{\partial \sigma_x}{\partial x} + b_x = 0$
- Boundary conditions:

$$u = \bar{u}\ (=0), \qquad x = a$$
$$t_x = \bar{t}_x = n_x\sigma_x\ (=0), \qquad x = b$$

Strong form

$$\frac{\partial}{\partial x}\left(E\frac{\partial u}{\partial x}\right) + b_x = 0$$

n_x is the unit outward normal vector

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2.2 Weak form of one-dimensional elasticity problems

■ **Weak form of equilibrium equation**

- We start by introducing **an arbitrary function $w(x)$** that is defined in the domain described by the interval $a < x < b$. Multiplying the equilibrium equation by this function we may write

$$\frac{\partial \sigma_x}{\partial x} + b_x = 0$$

$$g(\omega, u, \sigma_x) = w(x)\left(b_x + \frac{\partial \sigma_x}{\partial x}\right) = 0$$

- Integrate this product over the space domain of the problem

$$G(w, u, \sigma_x) = \int_\Omega w(x)\left(b_x + \frac{\partial \sigma_x}{\partial x}\right)\mathrm{d}x = 0$$

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2.2 Weak form of one-dimensional elasticity problems

■ **Weak form of equilibrium equation**

- Integrate the stress term by parts as

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

$$G(w, u, \sigma_x) = \int_\Omega w(x)\left(b_x + \frac{\partial \sigma_x}{\partial x}\right)\mathrm{d}x = 0$$

$$\Longrightarrow \quad \int_\Omega w(x)\frac{\partial \sigma_x}{\partial x}dx = w(x)n_x\sigma_x\big|_\Gamma - \int_\Omega \frac{\partial w}{\partial x}\sigma_x dx$$

- Where Γ is the boundary of Ω, and n_x is the outward pointing normal to the boundary. The boundary term may be expressed in terms of the traction as

$$w(x)n_x\sigma_x\big|_\Gamma = w(x)t_x(x)\big|_\Gamma = w(b)\sigma_x(b) - w(a)\sigma_x(a) = w(b)t_x(b) + w(a)t_x(a)$$

where we have noted $n_x(b) = 1$ and $n_x(a) = -1$.

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2.2 Weak form of one-dimensional elasticity problems

■ **Weak form of equilibrium equation**

- We also again introduce the notation that u is a boundary where $u = \bar{u}$, t is a boundary where $t_x = \bar{t}_x$, and the total boundary is $\Gamma = \Gamma_u \cup \Gamma_t$. With this notation we can write the *weak form* for equilibrium as

$$G(w, u, \sigma_x) = \int_\Omega w(x)\left(b_x + \frac{\partial \sigma_x}{\partial x}\right)\mathrm{d}x = 0$$

Weak form

⇩

$$G(w, u, \sigma_x) = \int_\Omega w(x)b_x\mathrm{d}x - \int_\Omega \frac{\partial w}{\partial x}\sigma_x\mathrm{d}x + wt_x\big|_{\Gamma_u} + w\bar{t}_x\big|_{\Gamma_t} = 0$$

⇩ $w\big|_{\Gamma_u} = 0$

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2.2 Weak form of one-dimensional elasticity problems

Weak form of equilibrium equation

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Weak form

⇩

$$G(w, u, \sigma_x) = \int_\Omega w(x) b_x \mathrm{d}x - \int_\Omega \frac{\partial w}{\partial x}\sigma_x \mathrm{d}x + w\bar{t}_x\big|_{\Gamma_t} = 0$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad ⇩ \quad \sigma_x = E\varepsilon_x$$

2.2 Weak form of one-dimensional elasticity problems

Weak form of equilibrium equation

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Weak form

⇩

$$G(w, u) = \int_\Omega w(x) b_x \mathrm{d}x - \int_\Omega \frac{\partial w}{\partial x} E \frac{\partial u}{\partial x}\mathrm{d}x + w\bar{t}_x\big|_{\Gamma_t} = 0$$

2.2 Finite element computation based on weak form

Galerkin method

- To construct an approximate solution we express the displacement $u(x)$ in terms of a set of specified functions multiplied by unknown parameters.

$$u(x) \approx \hat{u}(x) = \sum_{n=1}^{N} \phi_n(x) a_n + u_{\bar{\delta}}(x)$$

- In a similar way, we write the arbitrary function $w(x)$ in terms of an equal number of specified functions and arbitrary parameters. These may be expressed as

$$w(x) \approx \hat{w}(x) = \sum_{m=1}^{N} \psi_m(x) w_m$$

In the above form we assume that both $\phi_n(x)$ and $\psi_m(x)$ are zero at all locations where the boundary **DISPLACEMENT** is specified.

2.2 Finite element computation based on weak form

- The function $u_{\bar{\delta}}(x)$ is then specified as any function that satisfies the DISPLACEMENT boundary condition. For example, if the displacement must satisfy $u(L) = \bar{d}$ on the domain $0 < x < L$, this function may be taken as

$$u_{\bar{\delta}}(x) = \frac{x}{L}\bar{d} = \phi_{\bar{\delta}}(x)\bar{d}$$

2.2 Finite element computation based on weak form

■ **Galerkin method**

$$G(w,u)=\int_\Omega w(x)b_x\,\mathrm{d}x-\int_\Omega \frac{\partial w}{\partial x}E\frac{\partial u}{\partial x}\mathrm{d}x+w\bar{t}_x\Big|_{\Gamma_t}=0$$

$$u(x)\approx\hat{u}(x)=\sum_{n=1}^{N}\phi_n(x)a_n+u_\xi(x)\qquad u_\xi(x)=\frac{x}{L}\bar{d}=\phi_\xi(x)\bar{d}$$

$$w(x)\approx\hat{w}(x)=\sum_{m=1}^{N}\psi_m(x)w_m$$

- An approximate Galerkin solution for the elasticity problem

$$\hat{G}(\hat{w},\hat{u})=\sum_{m=1}^{N}w_m\int_\Omega\psi_m b_x\,\mathrm{d}x-\sum_{m=1}^{N}w_m\int_\Omega\frac{\mathrm{d}\psi_m}{\mathrm{d}x}E\left[\sum_{n=1}^{N}\frac{d\phi_n}{dx}a_n+\frac{d\phi_\xi}{dx}\bar{d}\right]\mathrm{d}x+\sum_{m=1}^{N}w_m\psi_m(x)\bar{t}_x\Big|_{\Gamma_t}=0$$

2.2 Finite element computation based on weak form

$$\hat{G}(\hat{w},\hat{u})=\sum_{m=1}^{N}w_m\int_\Omega\psi_m b_x\,\mathrm{d}x-\sum_{m=1}^{N}w_m\int_\Omega\frac{\mathrm{d}\psi_m}{\mathrm{d}x}E\left[\sum_{n=1}^{N}\frac{d\phi_n}{dx}a_n+\frac{d\phi_\xi}{dx}\bar{d}\right]\mathrm{d}x+\sum_{m=1}^{N}w_m\psi_m(x)\bar{t}_x\Big|_{\Gamma_t}=0$$

$$f_m=\int_\Omega\psi_m b_x\,\mathrm{d}x-\int_\Omega\frac{\mathrm{d}\psi_m}{\mathrm{d}x}E\frac{\mathrm{d}\phi_\xi}{\mathrm{d}x}\bar{d}\,\mathrm{d}x+\psi_m\bar{t}_x\Big|_{\Gamma_t}$$

load matrix

$$K_{mn}=\int_\Omega\frac{\mathrm{d}\psi_m}{\mathrm{d}x}E\frac{\mathrm{d}\phi_n}{\mathrm{d}x}\mathrm{d}x$$

stiffness matrix

2.2 Finite element computation based on weak form

- Since the parameters w_m are arbitrary, the expression multiplying each one must be zero. This leads to the set of equations:

$$\sum_{n=1}^{N}K_{mn}a_n=f_m,\quad m=1,2,\ldots,N$$

- The original problem of partial differential equations has been reduced to a set of algebraic equations.

2.2 Finite element computation based on weak form

- Matrix form of stiffness equation

$$\mathbf{Ka}=\mathbf{f}$$

$$\mathbf{K}=\begin{bmatrix}K_{11}&K_{12}&\cdots&K_{1N}\\K_{21}&K_{22}&&K_{1N}\\\vdots&&&\vdots\\K_{N1}&K_{N2}&\cdots&K_{NN}\end{bmatrix}\quad\text{and}\quad\mathbf{f}=\begin{Bmatrix}f_1\\f_2\\\vdots\\f_N\end{Bmatrix}$$

- Formal solution

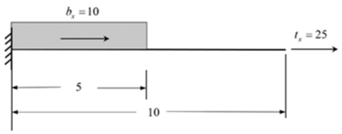
$$\mathbf{a}=\mathbf{K}^{-1}\mathbf{f}$$

- Displacement and stress solution

$$u(x)\approx\hat{u}(x)=\sum_{n=1}^{N}\phi_n(x)a_n+u_\xi(x)\qquad\hat{\sigma}_x(x)=E\frac{\partial\hat{u}(x)}{\partial x}=E\left(\sum_{n=1}^{N}\frac{\mathrm{d}\phi_n}{\mathrm{d}x}a_n+\frac{\mathrm{d}\phi_\xi(x)}{\mathrm{d}x}\bar{d}\right)$$

2.2 Finite element computation based on weak form

- **Example 3.1. Solutions of Galerkin method for one-dimensional elastic cantilever beam**
  - As an example we consider a static problem with length 10 units and $E = 1000$. Figure 3.1 shows the problem to solve.



- **Figure 3.1 One-dimensional elastic cantilever beam in Example 3.1.**

- Loading $b_x = \begin{cases} 10 & \text{for } 0 < x < 5 \\ 0 & \text{for } 5 < x < 10 \end{cases}$

- Boundary conditions

$$u(0) = 0 \quad \text{and} \quad \bar{t}_x(10) = 25$$

2.2 Finite element computation based on weak form

- Weak form

$$G(w, u) = \int_\Omega w(x) b_x \mathrm{d}x - \int_\Omega \frac{\partial w}{\partial x} E \frac{\partial u}{\partial x} \mathrm{d}x + w\bar{t}_x\big|_{\Gamma_t} = 0$$

$$\int_0^{10} \frac{\partial w}{\partial x} 1000 \frac{\partial u}{\partial x} \mathrm{d}x - \int_0^5 w(x) 10 \,\mathrm{d}x - w(10)25 = 0$$

- consider an approximate solution

$$u(x) \approx \hat{u}(x) = \sum_{n=1}^{N} \phi_n(x) a_n + u_{\bar{b}}(x) \qquad w(x) \approx \hat{w}(x) = \sum_{m=1}^{N} \psi_m(x) w_m$$

$$\hat{u}(x) = \sum_{n=1}^{N} \left(\frac{x}{10}\right)^n a_n \quad \text{and} \quad \hat{w}(x) = \sum_{m=1}^{N} \left(\frac{x}{10}\right)^m w_m$$

2.2 Finite element computation based on weak form

- Weak form

$$\sum_{n=1}^{N} \left[\int_0^{10} 10mn \left(\frac{x}{10}\right)^{m-1} \left(\frac{x}{10}\right)^{n-1} \mathrm{d}x \right] a_n = \int_0^5 \left(\frac{x}{10}\right)^m 10 \,\mathrm{d}x + 25$$

$$K_{mn} = \int_0^{10} 10mn \left(\frac{x}{10}\right)^{m+n-2} \mathrm{d}x = \frac{100mn}{m+n-1}$$

$$f_m = \int_0^5 \left(\frac{x}{10}\right)^m 10 \,\mathrm{d}x + 25 = \frac{100}{m+1} \left(\frac{1}{2}\right)^{m+1} + 25$$

2.2 Finite element computation based on weak form

- For example, M=N=2, the stiffness matrix and load matrix are given by

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} 100 & 100 \\ 100 & 400/3 \end{bmatrix}$$

$$\mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} 75/2 \\ 175/6 \end{Bmatrix}$$

- Matrix form of stiffness equation

$$\mathbf{Ka = f}$$

2.2 Treatments on boundary conditions

- Formal solution $\mathbf{a} = \mathbf{K}^{-1}\mathbf{f}$

$$\begin{bmatrix} 100 & 100 \\ 100 & 400/3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 75/2 \\ 175/6 \end{Bmatrix} \Rightarrow \mathbf{a} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0.62500 \\ -0.25000 \end{Bmatrix}$$

- Displacement and stress solution

$$u(x) \approx \hat{u}(x) = \sum_{n=1}^{N} \phi_n(x) a_n + u_5(x) \qquad \hat{\sigma}_x(x) = E\frac{\partial \hat{u}(x)}{\partial x} = E\left(\sum_{n=1}^{N} \frac{\mathrm{d}\phi_n}{\mathrm{d}x} a_n + \frac{\mathrm{d}\phi_5(x)}{\mathrm{d}x}\bar{d}\right)$$

$$u(x) \approx \hat{u}(x) = \sum_{n=1}^{2} \left(\frac{x}{10}\right)^n a_n = \frac{x}{10} \times 0.625 + \left(\frac{x}{10}\right)^2 \times (-0.25) = -0.0025x^2 + 0.0625x$$

$$\hat{\sigma}_x(x) = E\frac{\partial \hat{u}(x)}{\partial x} = E\left(\sum_{n=1}^{2} \frac{\mathrm{d}\phi_n}{\mathrm{d}x} a_n\right) = -5x + 62.5$$

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2.2 Finite element computation based on weak form

- Similarly, we get the solutions under different number of terms

N-terms	a_1	a_2	a_3	a_4	a_5
1	0.37500				
2	0.62500	-0.25000			
3	0.78125	-0.71875	0.31250		
4	0. 78125	-0.71875	0.31250	0.00000	
5	0.73437	-0.25000	-1.09275	1.64063	0.65625

- **Table 3.1 Parameters for one-dimensional elastic cantilever beam in Example 3.1.**

- Discard terms that do not contribute to the solution.
- Does not lead to convergent behavior.

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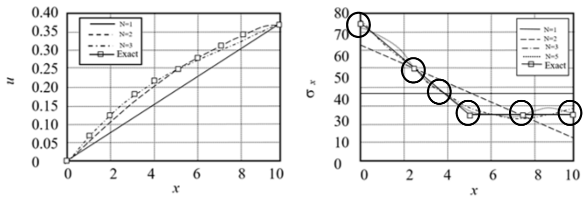
2.2 Finite element computation based on weak form

- Displacement and stress solution

1. The displacement at the free end is the same no matter how many terms we use. This often happens in one-dimensional static problems but, unfortunately, is seldom true in higher dimensional problems.
2. The solution for stress converges more slowly than that for the displacements;
3. however, once again we observe that some points are more accurate than others. These we shall call ***super-convergent*** **points** and these points play an important role in our later discussion on error estimates and adaptive refinement of solutions.



(a) (b)

- **Figure 3.2 Displacement and stress solutions in Example 3.1 based on Galerkin method using *N*-terms solutions: (a) displacement and (b) stress.**

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2.2 Finite element computation based on weak form

■ Finite element computation

- A more convenient method to construct the approximating functions ϕ_n and ψ_m are obtained by dividing the domain to be analyzed into small regular shaped regions. For example, we can divide the one-dimensional region between *a* and *b* into a set of "*M*" small finite segments by defining a set of *N* points x_i such that

$$x_1 = a, \quad x_i < x_{i+1} \quad \text{and} \quad x_N = b$$

For a one-dimensional problem we can let each increment define a *finite element* domain (or more simply, an *element*) and the set of points define the *nodes* (*finite element mesh* or *mesh*).

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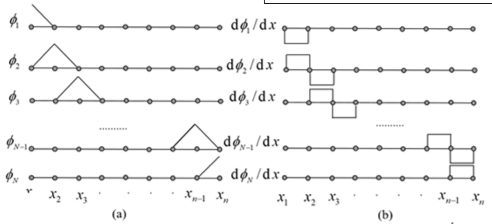
2.2 Finite element computation based on weak form

- A simple set of continuous polynomial approximating functions

$$\phi_i = \begin{cases} 0, & x < x_{i-1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i < x \le x_{i+1} \\ 0, & x > x_{i+1} \end{cases}$$

C_0 function since only the function is continuous in x, whereas the first derivative is only piecewise continuous with the discontinuities located at the nodes.



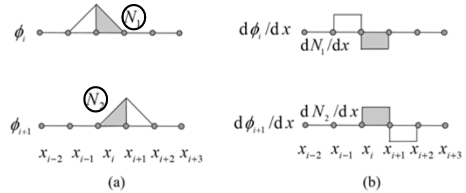
- **Figure 3.3 One-dimensional finite element approximation for ϕ_i : (a) functions and (b) derivatives.**

2.2 Finite element computation based on weak form

- Integrals over each element in the weak form functional

$$\int_\Omega (\cdot)\,\mathrm{d}x = \sum_{i=1}^{M} \int_{x_i}^{x_{i+1}} (\cdot)\,\mathrm{d}x \equiv \sum_e \int_{\Omega_e} (\cdot)\,\mathrm{d}x$$

Considering any interval $[x_i, x_{i+1}]$, we note that each interval is defined by the same two local functions N_1 and N_2. We call these the *shape functions* (形函数) for the element. To simplify the notation we also define local nodal coordinates on each element as x_1^e and x_2^e.



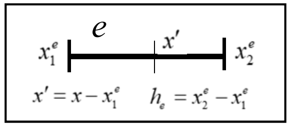
- **Figure 3.4 One-dimensional finite element shape functions: (a) functions and (b) derivatives.**

2.2 Finite element computation based on weak form

- Displacements and arbitrary weight function

$$\hat{u}^e = N_1(x')\hat{u}_1^e + N_2(x')\hat{u}_2^e$$
$$\hat{w}^e = N_1(x')\hat{w}_1^e + N_2(x')\hat{w}_2^e$$

- Local coordinate system



$$N_1(x') = 1 - \frac{x'}{h_e} \quad \text{and} \quad N_2(x') = \frac{x'}{h_e}$$

$$\frac{\mathrm{d}N_1}{\mathrm{d}x} = \frac{\mathrm{d}N_1}{\mathrm{d}x'} = -\frac{1}{h_e} \quad \text{and} \quad \frac{\mathrm{d}N_2}{\mathrm{d}x} = \frac{\mathrm{d}N_2}{\mathrm{d}x'} = \frac{1}{h_e}$$

2.2 Finite element computation based on weak form

- Weak form in global domain

$$G(w, u) = \int_\Omega w(x) b_x \mathrm{d}x - \int_\Omega \frac{\partial w}{\partial x} E \frac{\partial u}{\partial x} \mathrm{d}x + w\bar{t}_x\big|_{\Gamma_t} = 0$$

- Weak form in each element

$$\hat{G}(\hat{w}, \hat{u}) = \hat{G}_\sigma(\hat{w}, \hat{u}) - \hat{G}_f(\hat{w}, \hat{u}) - \hat{w}(x)\bar{t}_x\big|_{\Gamma_t}$$

$$\hat{G}_\sigma(\hat{w}, \hat{u}) = \sum_{e=1}^{M} \begin{bmatrix} \hat{w}_1^e & \hat{w}_2^e \end{bmatrix} \int_0^{h_e} \begin{Bmatrix} \dfrac{\mathrm{d}N_1}{\mathrm{d}x'} \\ \dfrac{\mathrm{d}N_2}{\mathrm{d}x'} \end{Bmatrix} E_e \begin{bmatrix} \dfrac{\mathrm{d}N_1}{\mathrm{d}x'} & \dfrac{\mathrm{d}N_2}{\mathrm{d}x'} \end{bmatrix} dx' \begin{Bmatrix} \hat{u}_1^e \\ \hat{u}_2^e \end{Bmatrix}$$

$$\hat{G}_f(\hat{w}, \hat{u}) = \sum_{e=1}^{M} \begin{bmatrix} \hat{w}_1^e & \hat{w}_2^e \end{bmatrix} \int_0^{h_e} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} b_x \,\mathrm{d}x'$$

2.2 Finite element computation based on weak form

- Each element can be evaluated as element stiffness matrix and load matrix

$$\mathbf{K}^e=\int_0^{h_e}\begin{Bmatrix}\dfrac{\mathrm{d}N_1}{\mathrm{d}x'}\\ \dfrac{\mathrm{d}N_2}{\mathrm{d}x'}\end{Bmatrix}E\begin{bmatrix}\dfrac{\mathrm{d}N_1}{\mathrm{d}x'} & \dfrac{\mathrm{d}N_2}{\mathrm{d}x'}\end{bmatrix}dx'=\begin{bmatrix}K_{11}^e & K_{12}^e\\ K_{21}^e & K_{22}^e\end{bmatrix}$$

$$\mathbf{f}^e=\int_0^{h_e}\begin{Bmatrix}N_1\\ N_2\end{Bmatrix}b_x\,\mathrm{d}x'=\begin{Bmatrix}f_1^e\\ f_2^e\end{Bmatrix}$$

- Element stiffness matrix and load matrix

E_e and b_x are constant $\quad N_1(x')=1-\dfrac{x'}{h_e}$ and $N_2(x')=\dfrac{x'}{h_e}$

$$\mathbf{K}^e=\frac{E_e}{h_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \text{ and } \mathbf{f}^e=\frac{1}{2}b_x h_e\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

2.2 Global assembly from one-dimensional elements

- In the one-dimensional beam model, *N*-1 elements ("*M*" small finite segments) and *N* nodes are used

- **Figure 3.5 Element and node numbers of one-dimensional problem.**

2.2 Global assembly from one-dimensional elements

- Then for each element we define the relationship of the local nodes to the global node number

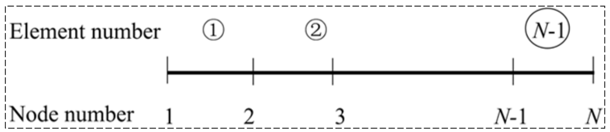
  - **Table 3.2 Local to global node numbering for two-end element for one-dimensional elastic beam.**

Local node number	Element number	1	2	3	⋯	*N*-1
	1	1	2	3	⋯	*N*-1
	2	2	3	4	⋯	*N*

2.2 Global assembly from one-dimensional elements

- According to the entire node number of each element, the corresponding element location vector can be provided for determining the relationships between local and global node locations as follows

$$\lambda^1=\{1\quad 2\}^T \qquad \lambda^2=\{2\quad 3\}^T \quad \cdot\cdot\cdot \quad \lambda^{N-1}=\{N-1\quad N\}^T$$



2.2 Global assembly from one-dimensional elements

- In order to implement global assembly from one-dimensional elements, the node numbers are marked on the left and upper sides of the matrix, and the expanded form of the stiffness matrix is given via element location vector

$$
\begin{array}{ll}
\text{Node} & \\
\text{number} &
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & \cdots & N-1 & N \end{array} \\
\begin{bmatrix}
K_{11}^{1} & K_{12}^{1} & 0 & & & \\
K_{21}^{1} & K_{22}^{1}+K_{11}^{2} & K_{12}^{2} & & 0 & \\
0 & K_{21}^{2} & (K_{22}^{2}+K_{11}^{3}) & & K_{12}^{3} & \\
 & & & & \ddots & \\
 & & & K_{21}^{N-1} & (K_{22}^{N-1}+K_{11}^{N}) & K_{12}^{N} \\
 & & & & K_{21}^{N} & K_{22}^{N}
\end{bmatrix}
\end{array}
\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \vdots \\ \hat{u}_{N-1} \\ \hat{u}_N \end{Bmatrix}
= \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-1} \\ f_N \end{Bmatrix}
$$

2.2 Global assembly from one-dimensional elements

- Similarly, the of node numbers is marked on the left of the load vector, and the expanded load is given by

$$
\begin{array}{l} \text{Node} \\ \text{number} \end{array}
\begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \end{array}
\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-1} \\ f_N \end{Bmatrix}
=
\begin{Bmatrix} f_1^{1} \\ f_2^{1}+f_1^{2} \\ f_2^{2}+f_1^{3} \\ \vdots \\ f_2^{N-2}+f_1^{N-1} \\ f_2^{N-1} \end{Bmatrix}
$$

2.2 Global assembly from one-dimensional elements

- A standard linear problem with the final stiffness equations is as follows

$$\mathbf{Ku} = \mathbf{f}$$

$$
\begin{array}{ll}
\text{Node} & \\
\text{number} &
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & \cdots & N-1 & N \end{array} \\
\begin{bmatrix}
K_{11}^{1} & K_{12}^{1} & 0 & & & \\
K_{21}^{1} & K_{22}^{1}+K_{11}^{2} & K_{12}^{2} & & 0 & \\
0 & K_{21}^{2} & (K_{22}^{2}+K_{11}^{3}) & & K_{12}^{3} & \\
 & & & & \ddots & \\
 & & & K_{21}^{N-1} & (K_{22}^{N-1}+K_{11}^{N}) & K_{12}^{N} \\
 & & & & K_{21}^{N} & K_{22}^{N}
\end{bmatrix}
\end{array}
\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \vdots \\ \hat{u}_{N-1} \\ \hat{u}_N \end{Bmatrix}
= \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-1} \\ f_N \end{Bmatrix}
$$

2.2 Global assembly from one-dimensional elements

- Matrix form of stiffness equation

$$\boxed{\mathbf{Ku} = \mathbf{f}}$$

$$
\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1N} \\ K_{21} & K_{22} & & K_{1N} \\ \vdots & & & \vdots \\ K_{N1} & K_{N2} & \cdots & K_{NN} \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{Bmatrix}
$$

- Formal solution

$$\boxed{\mathbf{u} = \mathbf{K}^{-1}\mathbf{f}}$$

- Displacement and stress solution

$$\hat{u}^e(x') = N_1(x')\hat{u}_1^e + N_2(x')\hat{u}_2^e \qquad \hat{\sigma}_x^e(x') = E\frac{\partial \hat{u}^e(x')}{\partial x'} = E\left(\frac{\mathrm{d}N_1(x')}{\mathrm{d}x'}\hat{u}_1^e + \frac{\mathrm{d}N_2(x')}{\mathrm{d}x'}\hat{u}_2^e\right)$$

2.2 Treatments on boundary conditions

- (1) Displacement boundary conditions $\hat{u}(x_a) \equiv \bar{u}_a$ $u(0) \equiv \bar{u}$ $\hat{u}_1 \equiv \bar{u}$

$$\begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 & & \\ 0 & K_{21}^2 & \left(K_{22}^2 + K_{11}^3\right) & K_{12}^3 & & \\ & & & \ddots & & \\ & & K_{21}^{N-1} & \left(K_{22}^{N-1} + K_{11}^N\right) & K_{12}^N \\ & & & K_{21}^N & K_{22}^N \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \vdots \\ \hat{u}_{N-1} \\ \hat{u}_N \end{Bmatrix} = \begin{Bmatrix} \bar{u}_1 \\ f_2 - K_{21}^1 \bar{u}_1 \\ f_3 \\ \vdots \\ f_{N-1} \\ f_N \end{Bmatrix}$$

This is equivalent to setting $w_1 \equiv 0$

- (2) Traction boundary conditions $x = L$

$f_N \to f_N + \bar{t}_x(L)$

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The End

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