5 Isoparametric Element and Numerical Integration

5.2 Numerical integration

- **5.2.3 Two-dimensional integration for rectangle element**
- Using the isoparametric form, the transformation of the integrand is accomplished as

$$\int_{y_1^e}^{y_2^e}\int_{x_1^e}^{x_2^e} f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)j(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

where $j(\xi,\eta)$ is the determinant of the Jacobian matrix described above.

5.2 Numerical integration

- With the above choice for the domain of the parent element is now convenient to evaluate integrals using Gaussian quadrature. Accordingly, we can integrate in two directions using

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)j_e(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\approx\sum_{n=1}^{N}\sum_{m=1}^{M} f(\xi_m,\eta_n)j_e(\xi_m,\eta_n)w_m w_n$$

where m denotes quadrature points in the ξ direction, and n quadrature points in the η direction. The location and weight for quadrature points in each direction are those given in Table 3.1 for one-dimensional integrations.

5.2 Numerical integration

- **Example 3.3.** For two-dimensional rectangle element with four nodes by numerical integration, compute the following integration $\int_{-1}^{1}\int_{-1}^{1}(\xi^2+\eta^2)\,\mathrm{d}\xi\,\mathrm{d}\eta$.

  In this case the integration is recovered using a two-point quadrature formula for two directions, respectively.

5.2 Numerical integration

- **Example 3.3.** For two-dimensional rectangle element with four nodes by numerical integration, compute the following integration $\int_{-1}^{1}\int_{-1}^{1}(\xi^2+\eta^2)\,\mathrm{d}\xi\,\mathrm{d}\eta$.

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1}(\xi^2+\eta^2)\,\mathrm{d}\xi\,\mathrm{d}\eta&\approx\sum_{n=1}^{2}\sum_{m=1}^{2} g(\xi_m,\eta_n)w_m w_n\\&=g(\xi_1,\eta_1)w_1w_1+g(\xi_1,\eta_2)w_1w_2\\&\quad+g(\xi_2,\eta_1)w_2w_1+g(\xi_2,\eta_2)w_2w_2\\&=g\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)\times1\times1+g\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\times1\times1\\&\quad+g\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)\times1\times1+g\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\times1\times1\\&=\frac{8}{3}\end{aligned}$$

5.2 Numerical integration

- It can be seen that this result is exactly the same as the result of direct integration as shown below

$$\int_{-1}^{1}\int_{-1}^{1}(\xi^2+\eta^2)\,\mathrm{d}\xi\,\mathrm{d}\eta=\int_{-1}^{1}\int_{-1}^{1}\xi^2\,\mathrm{d}\xi\,\mathrm{d}\eta+\int_{-1}^{1}\int_{-1}^{1}\eta^2\,\mathrm{d}\xi\,\mathrm{d}\eta=\frac{4}{3}+\frac{4}{3}=\frac{8}{3}$$

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5.2 Numerical integration

- **5.2.5 Three-dimensional integration for hexahedron element**

- Using the isoparametric form, the transformation of the integrand is accomplished as

$$\int_{z_1^e}^{z_2^e}\int_{y_1^e}^{y_2^e}\int_{x_1^e}^{x_2^e} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta,\zeta)j(\xi,\eta,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$$

where $j_e(\xi,\eta,\zeta)$ is the determinant of the Jacobian matrix described above.

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- Now the numerical integration form is

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta,\zeta)j(\xi,\eta,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \approx \sum_{l=1}^{L}\sum_{n=1}^{N}\sum_{m=1}^{M} f(\xi_m,\eta_n,\zeta_l)j(\xi_m,\eta_n,\zeta_l)w_m w_n w_l$$

where m denotes quadrature points in the ξ direction, n quadrature points in the η direction, and l quadrature points in the ζ direction. The location and weight for quadrature points in each direction are those given in Table 3.1 for one-dimensional integrations.

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- **Example 3.4.** For three-dimensional hexahedron element with eight nodes by numerical integration, compute the following integration $\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(\xi^2+\eta^2+\zeta^2)\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$.

  In this case the integration is recovered using a two-point quadrature formula for three directions, respectively.

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$$\begin{aligned}
&\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(\xi^2+\eta^2+\zeta^2)\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \approx \sum_{l=1}^{2}\sum_{n=1}^{2}\sum_{m=1}^{2} g(\xi_m,\eta_n,\zeta_l)w_m w_n w_l \\
&= g(\xi_1,\eta_1,\zeta_1)w_1w_1w_1 + g(\xi_1,\eta_1,\zeta_2)w_1w_1w_2 \\
&\quad + g(\xi_1,\eta_2,\zeta_1)w_1w_2w_1 + g(\xi_1,\eta_2,\zeta_2)w_1w_2w_2 \\
&\quad + g(\xi_2,\eta_1,\zeta_1)w_2w_1w_1 + g(\xi_2,\eta_1,\zeta_2)w_2w_1w_2 \\
&\quad + g(\xi_2,\eta_2,\zeta_1)w_2w_2w_1 + g(\xi_2,\eta_2,\zeta_2)w_2w_2w_2 \\
&= g\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)\times1\times1\times1 + g\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\times1\times1\times1 \\
&\quad + g\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)\times1\times1\times1 + g\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\times1\times1\times1 \\
&\quad + g\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)\times1\times1\times1 + g\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\times1\times1\times1 \\
&\quad + g\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)\times1\times1\times1 + g\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\times1\times1\times1 \\
&= 8
\end{aligned}$$

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5.2 Numerical integration

- It can be seen that this result is exactly the same as the result of direct integration as shown below

$$\begin{aligned}
&\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(\xi^2+\eta^2+\zeta^2)\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \\
&= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\xi^2\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\eta^2\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\zeta^2\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \\
&= \frac{8}{3}+\frac{8}{3}+\frac{8}{3} = 8
\end{aligned}$$

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- **5.2.6 Required order of numerical integration**

- For one-dimensional problem, with appropriate choice of the points ξ_j and weights w_j the formula integrates a polynomial in ξ of degree 2n−1 exactly.
- For two-dimensional and three-dimensional problems, the number of integration points in each dimension is not required to be the same, but for convenience, it is often taken as the same value.

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The End

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