6 Finite Element Computation Scheme of Elasticity Problems

6.1 Weak form for general elasticity problems

6.2 Finite element method for solving elasticity problems

6.3 Global assembly from high dimensional elements

6.4 Treatments on boundary conditions

6.5 Exercises

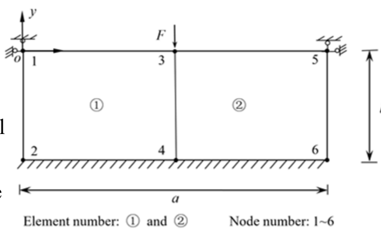
6.3 Global assembly from high dimensional elements

- The global assembly of two-dimensional or three-dimensional elements is exactly the same as that of the previous one-dimensional problem according to the element location vector.
- For the convenience of introduction, a typical two-dimensional example is given to explain the global assembly process and computation procedure.

6.3 Global assembly from high dimensional elements

- **Example 6.1.** Solutions of finite element method for two-dimensional plane strain problems.

The sizes of the model are a=30 in the horizontal x-direction and b=10 in the vertical y-direction. The material parameters are young's modulus E of 1×10^6 and Poisson's ratio v of 0.3.

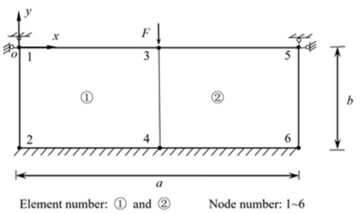


- **Figure 6.1 Element and node numbers of two-dimensional problem.**

6.3 Global assembly from high dimensional elements

- **Example 6.1.** Solutions of finite element method for two-dimensional plane strain problems.

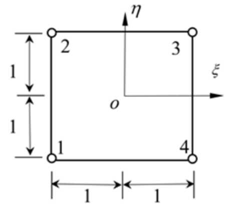
The boundary conditions are that the upper left and upper right vertices are fixed, and the lower parts are fixed. The vertical concentrated load F acting on the middle domain is -1×10^6.



- **Figure 6.1 Element and node numbers of two-dimensional problem.**

6.3 Global assembly from high dimensional elements

- Then for each element in local parent coordinate system, the standard two-dimensional rectangle element with four nodes as shown in Fig. 6.2 is used.

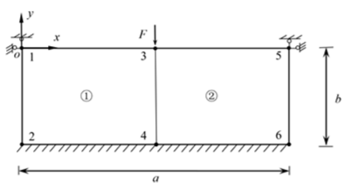


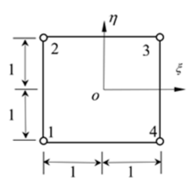
- **Figure 6.2 Two-dimensional rectangle element with four nodes in local parent coordinate system..**

6.3 Global assembly from high dimensional elements

- **Table 6.1 Local to global node numbering for two-dimensional problem.**

	Element number	1	2
Local node number	1	2	4
	2	1	3
	3	3	5
	4	4	6





6.3 Global assembly from high dimensional elements

- **Table 6.1 Local to global node numbering for two-dimensional problem.**

	Element number	1	2
Local node number	1	2	4
	2	1	3
	3	3	5
	4	4	6

- Element location vector can be provided for determining the relationships between local and global node locations

$$\lambda^1=\{2\quad 1\quad 3\quad 4\}^T\qquad \lambda^2=\{4\quad 3\quad 5\quad 6\}^T$$

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6.3 Global assembly from high dimensional elements

- In order to compute the global stiffness matrix, the stiffness matrix of each element should be computed according to the formula introduced in the previous section

$$\mathbf{K}^e_{ab}=\int_{\Omega_e}\mathbf{B}_a^T\mathbf{D}\mathbf{B}_b\,\mathrm{d}\Omega$$

- By substituting the corresponding parameters of this model, the stiffness matrices of the two elements shown below can be obtained. Here, the final computed results are given directly

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6.3 Global assembly from high dimensional elements

$$\mathbf{K}^e=\begin{bmatrix}\mathbf{K}^e_{11} & \mathbf{K}^e_{12} & \mathbf{K}^e_{13} & \mathbf{K}^e_{14}\\ \mathbf{K}^e_{21} & \mathbf{K}^e_{22} & \mathbf{K}^e_{23} & \mathbf{K}^e_{24}\\ \mathbf{K}^e_{31} & \mathbf{K}^e_{32} & \mathbf{K}^e_{33} & \mathbf{K}^e_{34}\\ \mathbf{K}^e_{41} & \mathbf{K}^e_{42} & \mathbf{K}^e_{43} & \mathbf{K}^e_{44}\end{bmatrix}$$

$$=\begin{bmatrix}491480.77 & 240384.60 & -42735.03 & -48076.92 & -245726.48 & -240384.60 & -202991.45 & 48076.92\\ & 758546.95 & 48076.92 & -630341.82 & -240384.60 & -379273.48 & -48076.92 & 251068.35\\ & & 491480.77 & -240384.60 & 202991.45 & -48076.92 & -245726.48 & 240384.60\\ & & & 758546.95 & 48076.92 & 251068.35 & 240384.60 & -379273.48\\ & & & & \boxed{491480.77} & 240384.60 & -42735.03 & -48076.92\\ & & & & & 758546.95 & 48076.92 & -630341.82\\ & \textit{Symmetric} & & & \underbrace{}_{\mathbf{K}^e_{33}} & & 491480.77 & -240384.60\\ & & & & & & & 758546.95\end{bmatrix}$$

- In order to explain the computation process of the above element stiffness matrix, the implementation process of the first coefficient 491480.77 in $\mathbf{K}^1_{33}$ of element 1 is introduced here.

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6.3 Global assembly from high dimensional elements

- Firstly, the matrix forms of the element stiffness matrix are expanded

$$\mathbf{K}^e_{ab}=\int_{\Omega_e}\mathbf{B}_a^T\mathbf{D}\mathbf{B}_b\,\mathrm{d}\Omega=\int_{\Omega_e}\mathbf{S}^T(N_a)\mathbf{D}\mathbf{S}^T(N_b)\mathrm{d}\Omega$$

$$=\int_{\Omega_e}\left\{\begin{bmatrix}\frac{\partial N_a}{\partial x} & 0 & 0 & \frac{\partial N_a}{\partial y}\\ 0 & \frac{\partial N_a}{\partial y} & 0 & \frac{\partial N_a}{\partial x}\end{bmatrix}\cdot\frac{E}{d}\begin{bmatrix}(1-v) & v & v & 0\\ v & (1-v) & v & 0\\ v & v & (1-v) & 0\\ 0 & 0 & 0 & (1-2v)/2\end{bmatrix}\begin{bmatrix}\frac{\partial N_b}{\partial x} & 0\\ 0 & \frac{\partial N_b}{\partial y}\\ 0 & 0\\ \frac{\partial N_b}{\partial y} & \frac{\partial N_b}{\partial x}\end{bmatrix}\right\}\mathrm{d}\Omega=\int_{\Omega_e}\left\{\frac{E}{d}\begin{bmatrix}\boxed{\frac{\partial N_a}{\partial x}\frac{\partial N_b}{\partial x}(1-v)+\frac{\partial N_a}{\partial y}\frac{\partial N_b}{\partial y}(1-2v)/2} & \frac{\partial N_a}{\partial x}\frac{\partial N_b}{\partial y}v+\frac{\partial N_a}{\partial y}\frac{\partial N_b}{\partial x}(1-2v)/2\\ \frac{\partial N_a}{\partial y}\frac{\partial N_b}{\partial x}v+\frac{\partial N_a}{\partial x}\frac{\partial N_b}{\partial y}(1-2v)/2 & \frac{\partial N_a}{\partial y}\frac{\partial N_b}{\partial y}(1-v)+\frac{\partial N_a}{\partial x}\frac{\partial N_b}{\partial x}(1-2v)/2\end{bmatrix}\right\}\mathrm{d}\Omega$$

where $d=(1+v)(1-2v)$.

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6.3 Global assembly from high dimensional elements

- It can be seen that the above formula involves the derivatives of shape functions to the global coordinates x and y, which can be converted to the local coordinate system ξ and η in the following form

$$\begin{Bmatrix}\frac{\partial N_a}{\partial x}\\ \frac{\partial N_a}{\partial y}\end{Bmatrix}=\mathbf{J}^{-1}\begin{Bmatrix}\frac{\partial N_a}{\partial \xi}\\ \frac{\partial N_a}{\partial \eta}\end{Bmatrix}=\frac{1}{j}\begin{bmatrix}\frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi}\\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi}\end{bmatrix}\begin{Bmatrix}\frac{\partial N_a}{\partial \xi}\\ \frac{\partial N_a}{\partial \eta}\end{Bmatrix}$$

$$j=\det\mathbf{J}=\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta}-\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}=(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}x_a)(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}y_a)-(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}x_a)(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}y_a)$$

$$=\frac{1}{\left((\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}x_a)(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}y_a)-(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}x_a)(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}y_a)\right)}\begin{bmatrix}\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}y_a & -\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}y_a\\ -\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}x_a & \sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}x_a\end{bmatrix}\begin{Bmatrix}\frac{\partial N_a}{\partial \xi}\\ \frac{\partial N_a}{\partial \eta}\end{Bmatrix}$$

$$=\frac{1}{\left((\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}x_a)(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}y_a)-(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}x_a)(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}y_a)\right)}\begin{bmatrix}(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}y_a)\frac{\partial N_a}{\partial \xi}-(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}y_a)\frac{\partial N_a}{\partial \eta}\\ -(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}x_a)\frac{\partial N_a}{\partial \xi}+(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}x_a)\frac{\partial N_a}{\partial \eta}\end{bmatrix}$$

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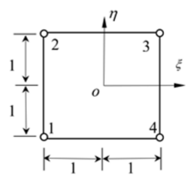
6.3 Global assembly from high dimensional elements

- Here it needs to use the following shape functions

$$N_1=\frac{1}{4}(1-\xi)\cdot(1-\eta),\quad N_2=\frac{1}{4}(1-\xi)\cdot(1+\eta)$$
$$N_3=\frac{1}{4}(1+\xi)\cdot(1+\eta),\quad N_4=\frac{1}{4}(1+\xi)\cdot(1-\eta)$$



- The derivatives of shape functions are

$$\frac{\partial N_1}{\partial \xi}=-\frac{1}{4}(1-\eta),\ \frac{\partial N_1}{\partial \eta}=-\frac{1}{4}(1-\xi),\quad \frac{\partial N_2}{\partial \xi}=-\frac{1}{4}(1+\eta),\ \frac{\partial N_2}{\partial \eta}=\frac{1}{4}(1-\xi)$$
$$\frac{\partial N_3}{\partial \xi}=\frac{1}{4}(1+\eta),\ \frac{\partial N_3}{\partial \eta}=\frac{1}{4}(1+\xi),\quad \frac{\partial N_4}{\partial \xi}=\frac{1}{4}(1-\eta),\ \frac{\partial N_4}{\partial \eta}=-\frac{1}{4}(1+\xi)$$

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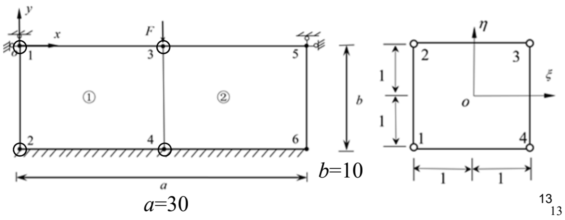
6.3 Global assembly from high dimensional elements

- Besides, the coordinate values of nodes in element 1 in the global coordinate system are

$$(x_1, y_1) = (0,-10),\ (x_2, y_2) = (0,0)$$
$$(x_3, y_3) = (15,0),\ (x_4, y_4) = (15,-10)$$



6.3 Global assembly from high dimensional elements

- By using the derivative of the shape function and the coordinate value of the node, it can be obtained

$$\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}x_a = -\frac{1}{4}(1-\eta)\cdot 0-\frac{1}{4}(1+\eta)\cdot 0+\frac{1}{4}(1+\eta)\cdot 15+\frac{1}{4}(1-\eta)\cdot 15=\frac{15}{2}$$
$$\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}y_a = -\frac{1}{4}(1-\xi)\cdot(-10)+\frac{1}{4}(1-\xi)\cdot 0+\frac{1}{4}(1+\xi)\cdot 0-\frac{1}{4}(1+\xi)\cdot(-10)=5$$
$$\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}x_a = -\frac{1}{4}(1-\xi)\cdot 0+\frac{1}{4}(1-\xi)\cdot 0+\frac{1}{4}(1+\xi)\cdot 15-\frac{1}{4}(1+\xi)\cdot 15=0$$
$$\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}y_a = -\frac{1}{4}(1-\eta)\cdot(-10)-\frac{1}{4}(1+\eta)\cdot 0+\frac{1}{4}(1+\eta)\cdot 0+\frac{1}{4}(1-\eta)\cdot(-10)=0$$

6.3 Global assembly from high dimensional elements

- Using the above formula to compute the derivative of the shape function N_3, and substituting the above results, we can get

$$\begin{Bmatrix}\dfrac{\partial N_3}{\partial x}\\ \dfrac{\partial N_3}{\partial y}\end{Bmatrix} = \frac{1}{\left((\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}x_a)(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}y_a)-(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}x_a)(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}y_a)\right)}\begin{bmatrix}(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}y_a)\frac{\partial N_3}{\partial \xi}-(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}y_a)\frac{\partial N_3}{\partial \eta}\\ -(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \eta}x_a)\frac{\partial N_3}{\partial \xi}+(\sum_{a=1}^{n}\frac{\partial N_a}{\partial \xi}x_a)\frac{\partial N_3}{\partial \eta}\end{bmatrix}$$
$$=\frac{1}{\left(\frac{15}{2}\cdot 5-0\cdot 0\right)}\begin{bmatrix}5\cdot\frac{1}{4}(1+\eta)-0\cdot\frac{1}{4}(1+\xi)\\ -0\cdot\frac{1}{4}(1+\eta)+\frac{15}{2}\frac{1}{4}(1+\xi)\end{bmatrix}=\frac{2}{75}\begin{bmatrix}\frac{5}{4}(1+\eta)\\ \frac{15}{8}(1+\xi)\end{bmatrix}=\begin{Bmatrix}\frac{1}{30}(1+\eta)\\ \frac{1}{20}(1+\xi)\end{Bmatrix}$$

The determinant of Jacobian matrix is $j=\dfrac{75}{2}$

6.3 Global assembly from high dimensional elements

- By substituting the derivatives of shape function N_3 and material parameters, the expression for the first coefficient in $\mathbf{K}_{33}^{1}$ of element 1 can be obtained

$$\mathbf{K}_{ab}^{e}=\int_{\Omega_e}\mathbf{B}_a^T\mathbf{D}\mathbf{B}_b\,\mathrm{d}\Omega=\int_{\Omega_e}\mathbf{S}^T(N_a)\mathbf{D}\mathbf{S}^T(N_b)\mathrm{d}\Omega$$
$$=\int_{\Omega_e}\left\{\begin{bmatrix}\frac{\partial N_a}{\partial x} & 0 & 0 & \frac{\partial N_a}{\partial y}\\ 0 & \frac{\partial N_a}{\partial y} & 0 & \frac{\partial N_a}{\partial x}\end{bmatrix}\cdot\frac{E}{d}\begin{bmatrix}(1-v) & v & v & 0\\ v & (1-v) & v & 0\\ v & v & (1-v) & 0\\ 0 & 0 & 0 & (1-2v)/2\end{bmatrix}\begin{bmatrix}\frac{\partial N_b}{\partial x} & 0\\ 0 & \frac{\partial N_b}{\partial y}\\ 0 & 0\\ \frac{\partial N_b}{\partial y} & \frac{\partial N_b}{\partial x}\end{bmatrix}\right\}\mathrm{d}\Omega=\int_{\Omega_e}\left\{\frac{E}{d}\begin{bmatrix}\boxed{\frac{\partial N_a}{\partial x}\frac{\partial N_b}{\partial x}(1-v)+\frac{\partial N_a}{\partial y}\frac{\partial N_b}{\partial y}(1-2v)/2} & \frac{\partial N_a}{\partial x}\frac{\partial N_b}{\partial y}v+\frac{\partial N_a}{\partial y}\frac{\partial N_b}{\partial x}(1-2v)/2\\ \frac{\partial N_a}{\partial y}\frac{\partial N_b}{\partial x}v+\frac{\partial N_a}{\partial x}\frac{\partial N_b}{\partial y}(1-2v)/2 & \frac{\partial N_a}{\partial y}\frac{\partial N_b}{\partial y}(1-v)+\frac{\partial N_a}{\partial x}\frac{\partial N_b}{\partial x}(1-2v)/2\end{bmatrix}\right\}\mathrm{d}\Omega$$

where $d=(1+v)(1-2v)$.

6.3 Global assembly from high dimensional elements

- By substituting the derivatives of shape function N_3 and material parameters, the expression for the first coefficient in $\mathbf{K}_{33}^{1}$ of element 1 can be obtained

$$\int_{\Omega_e}\frac{E}{d}\left[\frac{\partial N_3}{\partial x}\frac{\partial N_3}{\partial x}(1-v)+\frac{\partial N_3}{\partial y}\frac{\partial N_3}{\partial y}(1-2v)/2\right]\mathrm{d}\Omega$$
$$=\int_{\Omega_e}\frac{E}{(1+v)(1-2v)}\left[(\frac{1}{30})^2(1+\eta)^2(1-v)+(\frac{1}{20})^2(1+\xi)^2(1-2v)/2\right]j\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$=\int_{\Omega_e}\frac{10^6}{(1+0.3)(1-0.6)}\left[(\frac{1}{30})^2(1+\eta)^2\cdot 0.7+(\frac{1}{20})^2(1+\xi)^2\cdot 0.2\right]\frac{75}{2}\mathrm{d}\xi\,\mathrm{d}\eta$$
$$=1923076.92\int_{\Omega_e}\left[0.02917\cdot(1+\eta)^2+0.01875\cdot(1+\xi)^2\right]\mathrm{d}\xi\,\mathrm{d}\eta$$

6.3 Global assembly from high dimensional elements

- The integrand function in the above formula is a quadratic polynomial functions of ξ and η, so at least two integral points should be used in each direction, and there are four integral points in the element.

n	ξ_j	w_j
1	0	2.0
2	$\pm 1/\sqrt{3}$	1.0

$$\int_{\Omega_e}\frac{E}{d}\left[\frac{\partial N_3}{\partial x}\frac{\partial N_3}{\partial x}(1-v)+\frac{\partial N_3}{\partial y}\frac{\partial N_3}{\partial y}(1-2v)/2\right]\mathrm{d}\Omega$$
$$=1923076.92\int_{\Omega_e}\left[0.02917\cdot(1+\eta)^2+0.01875\cdot(1+\xi)^2\right]\mathrm{d}\xi\,\mathrm{d}\eta$$
$$=1923076.92\left[0.02917\cdot\left(1-\frac{1}{\sqrt{3}}\right)^2+0.01875\cdot\left(1-\frac{1}{\sqrt{3}}\right)^2\right]\cdot 1\cdot 1$$
$$\left[0.02917\cdot\left(1+\frac{1}{\sqrt{3}}\right)^2+0.01875\cdot\left(1-\frac{1}{\sqrt{3}}\right)^2\right]\cdot 1\cdot 1+\left[0.02917\cdot\left(1-\frac{1}{\sqrt{3}}\right)^2+0.01875\cdot\left(1+\frac{1}{\sqrt{3}}\right)^2\right]\cdot 1\cdot 1+$$
$$\left[0.02917\cdot\left(1+\frac{1}{\sqrt{3}}\right)^2+0.01875\cdot\left(1+\frac{1}{\sqrt{3}}\right)^2\right]\cdot 1\cdot 1$$
$$=491480.77$$

6.3 Global assembly from high dimensional elements

$$\mathbf{K}^e = \begin{bmatrix} \mathbf{K}^e_{11} & \mathbf{K}^e_{12} & \mathbf{K}^e_{13} & \mathbf{K}^e_{14} \\ \mathbf{K}^e_{21} & \mathbf{K}^e_{22} & \mathbf{K}^e_{23} & \mathbf{K}^e_{24} \\ \mathbf{K}^e_{31} & \mathbf{K}^e_{32} & \mathbf{K}^e_{33} & \mathbf{K}^e_{34} \\ \mathbf{K}^e_{41} & \mathbf{K}^e_{42} & \mathbf{K}^e_{43} & \mathbf{K}^e_{44} \end{bmatrix}$$

$$= \begin{bmatrix} 491480.77 & 240384.60 & -42735.03 & -48076.92 & -245726.48 & -240384.60 & -202991.45 & 48076.92 \\ & 758546.95 & 48076.92 & -630341.82 & -240384.60 & -379273.48 & -48076.92 & 251068.35 \\ & & 491480.77 & -240384.60 & 202991.45 & -48076.92 & -245726.48 & 240384.60 \\ & & & 758546.95 & 48076.92 & 251068.35 & 240384.60 & -379273.48 \\ & & & & \boxed{491480.77} & 240384.60 & -42735.03 & -48076.92 \\ & & & & & 758546.95 & 48076.92 & -630341.82 \\ & \mathit{Symmetric} & & & & & 491480.77 & -240384.60 \\ & & & & & & & 758546.95 \end{bmatrix}$$

- In this way, the stiffness coefficient can be obtained by numerical integration, further, other stiffness coefficients have the same computation process. Through the above steps, the values of stiffness matrices of each element can be obtained.

6.3 Global assembly from high dimensional elements

- In order to implement global assembly from two-dimensional elements, the node numbers are marked on the left and upper sides of the matrix, and the expanded form of the stiffness matrix is given via element location vector

$$\begin{array}{c} \text{Node} \\ \text{number } 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \mathbf{K}^1_{22} & \mathbf{K}^1_{21} & \mathbf{K}^1_{23} & \mathbf{K}^1_{24} & & \\ \mathbf{K}^1_{12} & \mathbf{K}^1_{11} & \mathbf{K}^1_{13} & \mathbf{K}^1_{14} & & \\ \mathbf{K}^1_{32} & \mathbf{K}^1_{31} & (\mathbf{K}^1_{33}+\mathbf{K}^2_{22}) & (\mathbf{K}^1_{34}+\mathbf{K}^2_{21}) & \mathbf{K}^2_{23} & \mathbf{K}^2_{24} \\ \mathbf{K}^1_{42} & \mathbf{K}^1_{41} & (\mathbf{K}^1_{43}+\mathbf{K}^2_{12}) & (\mathbf{K}^1_{44}+\mathbf{K}^2_{11}) & \mathbf{K}^2_{13} & \mathbf{K}^2_{14} \\ & & \mathbf{K}^2_{32} & \mathbf{K}^2_{31} & \mathbf{K}^2_{33} & \mathbf{K}^2_{34} \\ & & \mathbf{K}^2_{42} & \mathbf{K}^2_{41} & \mathbf{K}^2_{43} & \mathbf{K}^2_{44} \end{bmatrix} \end{array} \begin{Bmatrix} \hat{\mathbf{u}}_1 \\ \hat{\mathbf{u}}_2 \\ \hat{\mathbf{u}}_3 \\ \hat{\mathbf{u}}_4 \\ \hat{\mathbf{u}}_5 \\ \hat{\mathbf{u}}_6 \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{Bmatrix}$$

where **u** and **f** are global matrices of displacements and loads.

6.3 Global assembly from high dimensional elements

$$\lambda^1 = \{2 \quad 1 \quad 3 \quad 4\}^T \qquad \lambda^2 = \{4 \quad 3 \quad 5 \quad 6\}^T$$

$$\begin{array}{c} \text{Node} \\ \text{number } 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \mathbf{K}^1_{22} & \mathbf{K}^1_{21} & \mathbf{K}^1_{23} & \mathbf{K}^1_{24} & & \\ \mathbf{K}^1_{12} & \mathbf{K}^1_{11} & \mathbf{K}^1_{13} & \mathbf{K}^1_{14} & & \\ \mathbf{K}^1_{32} & \mathbf{K}^1_{31} & (\mathbf{K}^1_{33}+\mathbf{K}^2_{22}) & (\mathbf{K}^1_{34}+\mathbf{K}^2_{21}) & \mathbf{K}^2_{23} & \mathbf{K}^2_{24} \\ \mathbf{K}^1_{42} & \mathbf{K}^1_{41} & (\mathbf{K}^1_{43}+\mathbf{K}^2_{12}) & (\mathbf{K}^1_{44}+\mathbf{K}^2_{11}) & \mathbf{K}^2_{13} & \mathbf{K}^2_{14} \\ & & \mathbf{K}^2_{32} & \mathbf{K}^2_{31} & \mathbf{K}^2_{33} & \mathbf{K}^2_{34} \\ & & \mathbf{K}^2_{42} & \mathbf{K}^2_{41} & \mathbf{K}^2_{43} & \mathbf{K}^2_{44} \end{bmatrix} \end{array} \begin{Bmatrix} \hat{\mathbf{u}}_1 \\ \hat{\mathbf{u}}_2 \\ \hat{\mathbf{u}}_3 \\ \hat{\mathbf{u}}_4 \\ \hat{\mathbf{u}}_5 \\ \hat{\mathbf{u}}_6 \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{Bmatrix}$$

where **u** and **f** are global matrices of displacements and loads.

6.3 Global assembly from high dimensional elements

- The loads on the element can be obtained by integrating the product of the body forces and the shape functions

$$\mathbf{f}^e_a = \int_{\Omega_e} N_a \mathbf{b} \, \mathrm{d}\Omega \qquad \boxed{\mathbf{f}^e_a = \int_{\Omega_e} N_a \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_{et}} N_a \bar{\mathbf{t}} \, \mathrm{d}\Gamma}$$

- Similarly, the of node numbers is marked on the left of the load vector, and the expanded load is given via element location vector

$$\begin{array}{c} \text{Node } 1 \\ \text{number } 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^1_2 \\ \mathbf{f}^1_1 \\ \mathbf{f}^1_3 + \mathbf{f}^2_2 \\ \mathbf{f}^1_4 + \mathbf{f}^2_1 \\ \mathbf{f}^2_3 \\ \mathbf{f}^2_4 \end{Bmatrix} \qquad \begin{array}{l} \lambda^1 = \{2 \quad 1 \quad 3 \quad 4\}^T \\ \\ \lambda^2 = \{4 \quad 3 \quad 5 \quad 6\}^T \end{array}$$

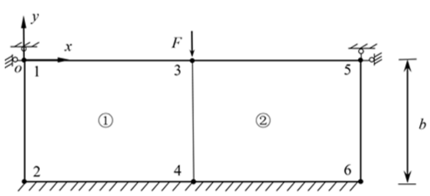
6 Finite Element Computation Scheme of Elasticity Problems

6.4 Treatments on boundary conditions

(1) Displacement boundary conditions

- With the above finite element form, it is very simple to impose the displacement boundary conditions since the parameters are now all physical values. That is they obey the property

$$\hat{\mathbf{u}}_1 = \hat{\mathbf{u}}_2 = \hat{\mathbf{u}}_4 = \hat{\mathbf{u}}_5 = \hat{\mathbf{u}}_6 \equiv \bar{\mathbf{u}} \equiv \mathbf{0}$$



6.4 Treatments on boundary conditions

- Thus, the stiffness equations can be expressed as

$$
\begin{array}{r}
\text{Node} \\
\text{number } 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \\
\mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & & \\
\mathbf{0} & \mathbf{0} & \left(\mathbf{K}_{33}^{1}+\mathbf{K}_{22}^{2}\right) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\
 & & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\
 & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}
\end{bmatrix}
\end{array}
\begin{Bmatrix} \hat{\mathbf{u}}_1 \\ \hat{\mathbf{u}}_2 \\ \hat{\mathbf{u}}_3 \\ \hat{\mathbf{u}}_4 \\ \hat{\mathbf{u}}_5 \\ \hat{\mathbf{u}}_6 \end{Bmatrix}
=
\begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{f}_3 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}
$$

where **I** is the identity matrix.

$$\boxed{\hat{\mathbf{u}}_1 = \hat{\mathbf{u}}_2 = \hat{\mathbf{u}}_4 = \hat{\mathbf{u}}_5 = \hat{\mathbf{u}}_6 \equiv \bar{\mathbf{u}} \equiv \mathbf{0}}$$

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6.4 Treatments on boundary conditions

- It should be noted that in order to simplify, the row and column in the stiffness matrix of known displacement on corresponding nodes can be deleted, which can reduce the scale of the stiffness matrix

$$\left(\mathbf{K}_{33}^{1}+\mathbf{K}_{22}^{2}\right)\hat{\mathbf{u}}_3 = \mathbf{f}_3$$

- The involved matrices and vectors are expanded, which can be expressed as

$$\begin{bmatrix} 982961.54 & 0 \\ 0 & 1517093.9 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} f_{3x} \\ f_{3y} \end{Bmatrix}$$

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6.3 Global assembly from high dimensional elements

$$\mathbf{K}^e = \begin{bmatrix} \mathbf{K}_{11}^e & \mathbf{K}_{12}^e & \mathbf{K}_{13}^e & \mathbf{K}_{14}^e \\ \mathbf{K}_{21}^e & \mathbf{K}_{22}^e & \mathbf{K}_{23}^e & \mathbf{K}_{24}^e \\ \mathbf{K}_{31}^e & \mathbf{K}_{32}^e & \mathbf{K}_{33}^e & \mathbf{K}_{34}^e \\ \mathbf{K}_{41}^e & \mathbf{K}_{42}^e & \mathbf{K}_{43}^e & \mathbf{K}_{44}^e \end{bmatrix}$$

$$= \begin{bmatrix}
491480.77 & 240384.60 & -42735.03 & -48076.92 & -245726.48 & -240384.60 & -202991.45 & 48076.92 \\
 & 758546.95 & 48076.92 & -630341.82 & -240384.60 & -379273.48 & -48076.92 & 251068.35 \\
 & & 491480.77 & -240384.60 & 202991.45 & -48076.92 & -245726.48 & 240384.60 \\
 & & & 758546.95 & 48076.92 & 251068.35 & 240384.60 & -379273.48 \\
 & & & & \boxed{491480.77} & 240384.60 & -42735.03 & -48076.92 \\
 & & & & & 758546.95 & 48076.92 & -630341.82 \\
 & \textit{Symmetric} & & & & & 491480.77 & -240384.60 \\
 & & & & & & & 758546.95
\end{bmatrix}$$

- In this way, the stiffness coefficient can be obtained by numerical integration, further, other stiffness coefficients have the same computation process. Through the above steps, the values of stiffness matrices of each element can be obtained.

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6.4 Treatments on boundary conditions

(2) Traction boundary conditions

- In order to solve the above matrix equation, we need to know the load on node 3. Since there is no body force at present, the load matrix is

$$\begin{Bmatrix} f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\boxed{\mathbf{f}_a^e = \int_{\Omega_e} N_a \mathbf{b} \, \mathrm{d}\Omega}$$

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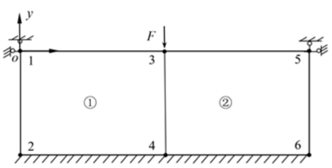
6.4 Treatments on boundary conditions

- Furthermore, the force acting on the boundary is considered. Imposing a traction condition at node 3 only requires the modification

$$f_{3y} \rightarrow f_{3y} + \bar{t} = -1\times10^{6}$$



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6.4 Treatments on boundary conditions

- The stiffness equation to be solved becomes the following form

$$\begin{bmatrix} 982961.54 & 0 \\ 0 & 1517093.9 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1000000 \end{Bmatrix}$$

- It is solved directly and the following results can be obtained

$$\begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.6592 \end{Bmatrix}$$

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6.4 Treatments on boundary conditions

- The displacement functions of each element can be obtained by the combination of shape function and node displacements

Element 1

$$\hat{u}^1(\xi,\eta)=\sum_{a=1}^{4}N_a(\xi,\eta)\,\hat{u}_a^1=N_1\cdot 0+N_2\cdot 0+N_3\cdot 0+N_4\cdot 0=0$$

$$\hat{v}^1(\xi,\eta)=\sum_{a=1}^{4}N_a(\xi,\eta)\,\hat{v}_a^1=N_1\cdot 0+N_2\cdot 0+N_3\cdot(-0.6592)+N_4\cdot 0$$
$$=-0.1648(1+\xi)(1+\eta)$$

6.4 Treatments on boundary conditions

- The displacement functions of each element by global coordinates

$$(x_1,y_1)=(0,-10),\ (x_2,y_2)=(0,0)$$
$$(x_3,y_3)=(15,0),\ (x_4,y_4)=(15,-10)$$

Element 1

$$x=\sum_{a=1}^{n}N_a x_a=N_1\cdot x_1-N_2\cdot x_2+N_3\cdot x_3+N_4\cdot x_4=\frac{15}{2}(1+\xi) \quad \Rightarrow \quad \xi=\frac{2}{15}x-1$$

$$y=\sum_{a=1}^{n}N_a y_a=N_1\cdot y_1-N_2\cdot y_2+N_3\cdot y_3+N_4\cdot y_4=-5(1-\eta) \qquad \eta=\frac{1}{5}y+1$$

$$\hat{u}^1(\xi,\eta)=\sum_{a=1}^{4}N_a(\xi,\eta)\,\hat{u}_a^1=N_1\cdot 0+N_2\cdot 0+N_3\cdot 0+N_4\cdot 0=0$$

$$\hat{v}^1(\xi,\eta)=\sum_{a=1}^{4}N_a(\xi,\eta)\,\hat{v}_a^1=N_1\cdot 0+N_2\cdot 0+N_3\cdot(-0.6592)+N_4\cdot 0$$
$$=-0.1648(1+\xi)(1+\eta)=-0.004395\,x(y+10)$$

6.4 Treatments on boundary conditions

- The displacement functions of each element can be obtained by the combination of shape function and node displacements

Element 2

$$\hat{u}^2(\xi,\eta)=\sum_{a=1}^{4}N_a(\xi,\eta)\,\hat{u}_a^2=N_1\cdot 0+N_2\cdot 0+N_3\cdot 0+N_4\cdot 0=0$$

$$\hat{v}^2(\xi,\eta)=\sum_{a=1}^{4}N_a(\xi,\eta)\,\hat{v}_a^2=N_1\cdot 0+N_2\cdot(-0.6592)+N_3\cdot 0+N_4\cdot 0$$
$$=-0.1648(1-\xi)(1+\eta)$$

6.4 Treatments on boundary conditions

- Once the displacement are known, the approximation for stresses on element may be computed by

$$\hat{\boldsymbol{\sigma}}=\sum_{b=1}^{4}\mathbf{DS}(N_b)\,\hat{\mathbf{u}}_b^e=\sum_{b=1}^{4}\mathbf{DB}_b\hat{\mathbf{u}}_b^e$$

- In the computation process, the derivatives of each shape function is needed, which is the same as the way for N_3 above. The final derivative results of other shape functions are given below

$$\begin{Bmatrix}\dfrac{\partial N_1}{\partial x}\\ \dfrac{\partial N_1}{\partial y}\end{Bmatrix}=\begin{Bmatrix}-\dfrac{1}{30}(1-\eta)\\ -\dfrac{1}{20}(1-\xi)\end{Bmatrix}\quad \begin{Bmatrix}\dfrac{\partial N_2}{\partial x}\\ \dfrac{\partial N_2}{\partial y}\end{Bmatrix}=\begin{Bmatrix}-\dfrac{1}{30}(1+\eta)\\ \dfrac{1}{20}(1-\xi)\end{Bmatrix}\quad \begin{Bmatrix}\dfrac{\partial N_4}{\partial x}\\ \dfrac{\partial N_4}{\partial y}\end{Bmatrix}=\begin{Bmatrix}\dfrac{1}{30}(1-\eta)\\ -\dfrac{1}{20}(1+\xi)\end{Bmatrix}$$

6.4 Treatments on boundary conditions

- The stress functions of each element can be obtained by the combination of derivatives of shape functions and node displacements

Element 1

$$\hat{\boldsymbol{\sigma}}^1=\{\sigma_x^1\ \ \sigma_y^1\ \ \sigma_z^1\ \ \tau_{xy}^1\}^T=\sum_{b=1}^{4}\mathbf{DS}(N_b)\hat{\mathbf{u}}_b^1=\sum_{b=1}^{4}\mathbf{DB}_b\hat{\mathbf{u}}_b^1$$
$$=\mathbf{D}(\mathbf{B}_1\hat{\mathbf{u}}_1^1+\mathbf{B}_2\hat{\mathbf{u}}_2^1+\mathbf{B}_3\hat{\mathbf{u}}_3^1+\mathbf{B}_4\hat{\mathbf{u}}_4^1)=\mathbf{D}(\mathbf{B}_1 0+\mathbf{B}_2 0+\mathbf{B}_3\hat{\mathbf{u}}_3^1+\mathbf{B}_4 0)=\mathbf{DB}_3\hat{\mathbf{u}}_3^1$$
$$=\frac{E}{d}\begin{bmatrix}(1-v)&v&v&0\\ v&(1-v)&v&0\\ v&v&(1-v)&0\\ 0&0&0&(1-2v)/2\end{bmatrix}\begin{bmatrix}\dfrac{\partial N_3}{\partial x}&0\\ 0&\dfrac{\partial N_3}{\partial y}\\ 0&0\\ \dfrac{\partial N_3}{\partial y}&\dfrac{\partial N_3}{\partial x}\end{bmatrix}\begin{Bmatrix}\hat{u}_3^1\\ \hat{v}_3^1\end{Bmatrix}$$
$$=\frac{E}{d}\begin{bmatrix}(1-v)&v&v&0\\ v&(1-v)&v&0\\ v&v&(1-v)&0\\ 0&0&0&(1-2v)/2\end{bmatrix}\begin{bmatrix}\frac{1}{30}(1+\eta)&0\\ 0&\frac{1}{20}(1+\xi)\\ 0&0\\ \frac{1}{20}(1+\xi)&\frac{1}{30}(1+\eta)\end{bmatrix}\begin{Bmatrix}0\\ -0.6592\end{Bmatrix}$$

6.4 Treatments on boundary conditions

- The stress functions of each element can be obtained by the combination of derivatives of shape functions and node displacements

Element 1

$$\hat{\boldsymbol{\sigma}}^1=\frac{E}{d}\begin{bmatrix}\frac{1}{30}(1+\eta)(1-v)&\frac{1}{20}(1+\xi)v\\ \frac{1}{30}(1+\eta)v&\frac{1}{20}(1+\xi)(1-v)\\ \frac{1}{30}(1+\eta)v&\frac{1}{20}(1+\xi)v\\ \frac{1}{20}(1+\xi)(1-2v)/2&\frac{1}{30}(1+\eta)(1-2v)/2\end{bmatrix}\begin{Bmatrix}0\\ -0.6592\end{Bmatrix}$$
$$=-0.6592\frac{E}{d}\begin{Bmatrix}\frac{1}{20}(1+\xi)v\\ \frac{1}{20}(1+\xi)(1-v)\\ \frac{1}{20}(1+\xi)v\\ \frac{1}{30}(1+\eta)(1-2v)/2\end{Bmatrix}=-126769231\begin{Bmatrix}0.15(1+\xi)\\ 0.35(1+\xi)\\ 0.15(1+\xi)\\ 0.067(1+\eta)\end{Bmatrix}=\begin{Bmatrix}-1901538(1+\xi)\\ -4436923(1+\xi)\\ -1901538(1+\xi)\\ -8451.28(1+\eta)\end{Bmatrix}$$

6.4 Treatments on boundary conditions

- The stress functions of each element can be obtained by the combination of derivatives of shape functions and node displacements

Element 2 $\hat{\boldsymbol{\sigma}}^2 = \{\sigma_x^2 \quad \sigma_y^2 \quad \sigma_z^2 \quad \tau_{xy}^2\}^T = \sum_{b=1}^{4} \mathbf{DS}(N_b)\hat{\mathbf{u}}_b^2 = \sum_{b=1}^{4} \mathbf{DB}_b\hat{\mathbf{u}}_b^2$

$$= \mathbf{D}(\mathbf{B}_1\hat{\mathbf{u}}_1^2 + \mathbf{B}_2\hat{\mathbf{u}}_2^2 + \mathbf{B}_3\hat{\mathbf{u}}_3^2 + \mathbf{B}_4\hat{\mathbf{u}}_4^2) = \mathbf{D}(\mathbf{B}_1 0 + \mathbf{B}_2\hat{\mathbf{u}}_2^2 + \mathbf{B}_3 0 + \mathbf{B}_4 0) = \mathbf{DB}_2\hat{\mathbf{u}}_2^2$$

$$= \frac{E}{d}\begin{bmatrix} (1-v) & v & v & 0 \\ v & (1-v) & v & 0 \\ v & v & (1-v) & 0 \\ 0 & 0 & 0 & (1-2v)/2 \end{bmatrix} \begin{bmatrix} \frac{\partial N_2}{\partial x} & 0 \\ 0 & \frac{\partial N_2}{\partial y} \\ 0 & 0 \\ \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} \end{bmatrix} \begin{Bmatrix} \hat{u}_2^2 \\ \hat{v}_2^2 \end{Bmatrix}$$

$$= \frac{E}{d}\begin{bmatrix} (1-v) & v & v & 0 \\ v & (1-v) & v & 0 \\ v & v & (1-v) & 0 \\ 0 & 0 & 0 & (1-2v)/2 \end{bmatrix} \begin{bmatrix} -\frac{1}{30}(1+\eta) & 0 \\ 0 & \frac{1}{20}(1-\xi) \\ 0 & 0 \\ \frac{1}{20}(1-\xi) & -\frac{1}{30}(1+\eta) \end{bmatrix} \begin{Bmatrix} 0 \\ -0.6592 \end{Bmatrix}$$

6.4 Treatments on boundary conditions

- The stress functions of each element can be obtained by the combination of derivatives of shape functions and node displacements

Element 2

$$\hat{\boldsymbol{\sigma}}^2 = \frac{E}{d}\begin{bmatrix} -\frac{1}{30}(1+\eta)(1-v) & \frac{1}{20}(1-\xi)v \\ -\frac{1}{30}(1+\eta)v & \frac{1}{20}(1-\xi)(1-v) \\ -\frac{1}{30}(1+\eta)v & \frac{1}{20}(1-\xi)v \\ \frac{1}{20}(1-\xi)(1-2v)/2 & -\frac{1}{30}(1+\eta)(1-2v)/2 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.6592 \end{Bmatrix}$$

$$= -0.6592\frac{E}{d}\begin{Bmatrix} \frac{1}{20}(1-\xi)v \\ \frac{1}{20}(1-\xi)(1-v) \\ \frac{1}{20}(1-\xi)v \\ -\frac{1}{30}(1+\eta)(1-2v)/2 \end{Bmatrix} = -126769.231\begin{Bmatrix} 0.15(1-\xi) \\ 0.35(1-\xi) \\ 0.15(1-\xi) \\ -0.067(1+\eta) \end{Bmatrix} = \begin{Bmatrix} -19015.38(1-\xi) \\ -44369.23(1-\xi) \\ -19015.38(1-\xi) \\ 8451.28(1+\eta) \end{Bmatrix}$$

6.4 Treatments on boundary conditions

- Through the solutions of the above stress functions, the stress values at the integration points can be obtained, by substituting the coordinates of the integration points of the specified element. For example, the stress at integral point $(\xi, \eta) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ of element 1 is

$$\hat{\boldsymbol{\sigma}}^1 = \{\sigma_x^1 \quad \sigma_y^1 \quad \sigma_z^1 \quad \tau_{xy}^1\}^T = \begin{Bmatrix} -19015.38(1+\xi) \\ -44369.23(1+\xi) \\ -19015.38(1+\xi) \\ -8451.28(1+\eta) \end{Bmatrix} = \begin{Bmatrix} -19015.38\left(1-\frac{1}{\sqrt{3}}\right) \\ -44369.23\left(1-\frac{1}{\sqrt{3}}\right) \\ -19015.38\left(1-\frac{1}{\sqrt{3}}\right) \\ -8451.28\left(1+\frac{1}{\sqrt{3}}\right) \end{Bmatrix} = \begin{Bmatrix} -8036.84 \\ -18752.63 \\ -8036.84 \\ -13330.63 \end{Bmatrix}$$

6.4 Treatments on boundary conditions

- Furthermore, by substituting the node coordinates of the specified element into the solution of the above stress function, the stress values at the nodes can be obtained. For example, the stresses at node 2 of $(\xi, \eta) = (-1, 1)$ of element 1 are as follows

$$\hat{\boldsymbol{\sigma}}^1 = \{\sigma_x^1 \quad \sigma_y^1 \quad \sigma_z^1 \quad \tau_{xy}^1\}^T = \begin{Bmatrix} -19015.38(1+\xi) \\ -44369.23(1+\xi) \\ -19015.38(1+\xi) \\ -8451.28(1+\eta) \end{Bmatrix} = \begin{Bmatrix} -19015.38(1-1) \\ -44369.23(1-1) \\ -19015.38(1-1) \\ -8451.28(1+1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -16902.56 \end{Bmatrix}$$

The End