Computational Mechanics

(*English Course*)

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2 Fundamentals of Elastic Mechanics

- **Contents**

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- **Keywords**

- **Three-dimensional problem** 三维问题
- **Two-dimensional problem** 二维问题
- **Plane stress problem** 平面应力问题
- **Plane strain problem** 平面应变问题
- **Axisymmetric problem** 轴对称问题
- **Displacement** 位移 **Strain** 应变 **Stress** 应力
- **Geometric equations** 几何方程
- **Constitutive equations** 本构方程
- **Equilibrium equations** 平衡方程
- **Boundary conditions** 边界条件

2 Fundamentals of Elastic Mechanics

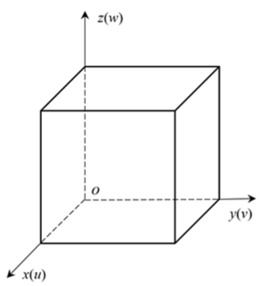
- **Basic variables and equations**

- The basic equations for the theory of elasticity are described in variables of **displacements, strains, stresses,** involving the equations of **geometric equations, constitutive equations, equilibrium equations, and boundary equations**.

- We start by specifying each equation set for a general three-dimensional problem in Cartesian coordinates. However, we will also consider the two-dimensional problems: ***plane stress***, ***plane strain*** and ***axisymmetric cases.***

2 Fundamentals of Elastic Mechanics

- **Three-dimensional problems**



- **Figure 2.1 Three-dimensional elasticity problems.**

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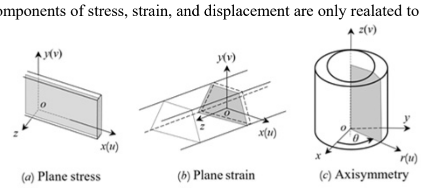
- **Two-dimensional problems**

(1) **Plane stress.** There is only nonzero stress in the problem plane here and no stress in the direction orthogonal to the thin plate, as exhibited in Figure 2.2(a)..

(2) **Plane strain.** It is assumed that the strain perpendicular to the plane under consideration is zero. This may occur in a prism, as shown in Figure 2.2(b), where the load perpendicular to the plane remains unchanged.

(3) **Axisymmetric.** In the cylindrical coordinate system *r–z–θ*, the angle θ in the plane considered is constant, as displayed in Figure 2.2 (c). It is assumed that all components of stress, strain, and displacement are only realated to *r* and *z*.



- **Figure 2.2 Two-dimensional elasticity problems.**

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2.1 Displacements

■ **Displacement function**

- **Three-dimensional problem**

$$\mathbf{u}(\mathbf{x})=\begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} \qquad \mathbf{x}=\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

- **Two-dimensional problem**

✓ ***plane stress and plane strain*** cases

$$\mathbf{u}(\mathbf{x})=\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$$

✓ ***axisymmetric*** case

$$\mathbf{u}(\mathbf{x})=\begin{Bmatrix} u(r,z) \\ v(r,z) \end{Bmatrix} \qquad \mathbf{x}=\begin{Bmatrix} r \\ z \end{Bmatrix}$$

2 Fundamentals of Elastic Mechanics

2.1 Displacements
2.2 Strains
2.3 Stresses
2.4 Geometric equations
2.5 Constitutive equations
2.6 Equilibrium equations
2.7 Boundary conditions
2.8 Exercises

2.2 Strains

- There are six independent strain components in a three-dimensional problem. They are arranged in order and expressed in the form of a matrix, that is:

$$\boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon_x & \varepsilon_y & \varepsilon_z & \gamma_{xy} & \gamma_{yz} & \gamma_{zx}\end{bmatrix}^T$$

This form is called Voigt notation. It is a way of writing a symmetric second order tensor in terms of a reduced set of components. The strain is a symmetric form where $\gamma_{xy}=\gamma_{yx}$, $\gamma_{yz}=\gamma_{zy}$, and $\gamma_{zx}=\gamma_{xz}$; thus, Voigt notation reduces nine components to six.

- For the two-dimensional problems, the last two components are always zero. Thus, only four components of **ε** need be considered.

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2.1 Displacements
2.2 Strains
2.3 Stresses
2.4 Geometric equations
2.5 Constitutive equations
2.6 Equilibrium equations
2.7 Boundary conditions
2.8 Exercises

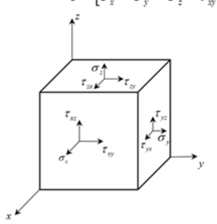
2.3 Stresses

- The components σ_x, σ_y, σ_z are called normal stresses and τ_{xy}, τ_{yx}, τ_{yz}, τ_{zy}, τ_{xz}, τ_{zx} are called shear stresses

$$\tau_{xy}=\tau_{yx},\ \tau_{yz}=\tau_{zy} \quad \text{and} \quad \tau_{zx}=\tau_{xz}$$

Thus, similar to strain, the stresses may be written in terms of six components that are ordered and denoted in matrix form by

$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma_x & \sigma_y & \sigma_z & \tau_{xy} & \tau_{yz} & \tau_{zx}\end{bmatrix}^T$$



- **Figure 2.3 Stresses in three-dimensional solid analysis.**

2.3 Stresses

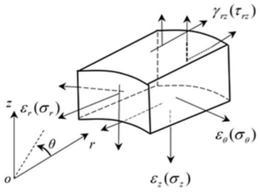
- For two-dimensional plane problems we consider only four components of stress and use

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \tau_{xy} \end{bmatrix}^T$$

In axisymmetry we define the components of stress as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_r & \sigma_z & \sigma_\theta & \tau_{rz} \end{bmatrix}^T$$

In plane stress problems we know a priori that $\sigma_z = 0$.



- **Figure 2.4 Stresses in axisymmetric solid analysis.**

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2.4 Geometric equations

■ **Strain and displacement relations**

- The strains for a problem undergoing small deformations are computed from the displacements and may be expressed in matrix form as

$$\boldsymbol{\varepsilon} = \mathbf{S}\mathbf{u}$$

where **S** is a matrix of differential operators and **u** is the displacement field. For the three-dimensional problem the strain-displacement relation is expressed as

■ **Three-dimensional problems**

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

2.4 Geometric equations

■ **Two-dimensional problems**

- For convenience in considering all three classes of two-dimensional problems in a unified manner, we include four components of strain in **ε** and write them as

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \varepsilon_z \\ 0 \end{Bmatrix} = \mathbf{S}_p\mathbf{u} + \boldsymbol{\varepsilon}_z$$

for plane problems (where $\boldsymbol{\varepsilon}_z$ is zero for plane strain but not for plane stress) and axisymmetric case

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = \mathbf{S}_a\mathbf{u}$$

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2.5 Constitutive equations

■ **Stress and strain relations: Elasticity matrix**

- Using this hypothesis the stress-strain equations for a linearly elastic material may be expressed by

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

or by

$$\boldsymbol{\varepsilon} = \mathbf{D}^{-1}\boldsymbol{\sigma}$$

The **D** matrix is known as the elasticity matrix of moduli and the $\mathbf{D}^{-1}$ matrix as the elasticity matrix of compliances (Inverse matrix).

2.5 Constitutive equations

- **Three-dimensional problems**
  - In cartesian coordinates system

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -v & -v & 0 & 0 & 0\\ -v & 1 & -v & 0 & 0 & 0\\ -v & -v & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+v) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+v) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+v)\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix}$$

Inverting to obtain the appropriate elasticity matrix of moduli yields the result

$$\mathbf{D}=\frac{E}{d}\begin{bmatrix}(1-v) & v & v & 0 & 0 & 0\\ v & (1-v) & v & 0 & 0 & 0\\ v & v & (1-v) & 0 & 0 & 0\\ 0 & 0 & 0 & (1-2v)/2 & 0 & 0\\ 0 & 0 & 0 & 0 & (1-2v)/2 & 0\\ 0 & 0 & 0 & 0 & 0 & (1-2v)/2\end{bmatrix}$$

$d=(1+v)(1-2v)$

$-1<v<\frac{1}{2}$

2.5 Constitutive equations

- **Two-dimensional problems**
  - Two-dimensional problems in Cartesian coordinates system

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\end{Bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -v & -v & 0\\ -v & 1 & -v & 0\\ -v & -v & 1 & 0\\ 0 & 0 & 0 & 2(1+v)\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\end{Bmatrix}$$

✓ The plane stress case

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\end{Bmatrix}=\frac{E}{(1-v^2)}\begin{bmatrix}1 & v & 0 & 0\\ v & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & (1-v)/2\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\end{Bmatrix}$$

✓ The plane strain case

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\end{Bmatrix}=\frac{E}{d}\begin{bmatrix}(1-v) & v & v & 0\\ v & (1-v) & v & 0\\ v & v & (1-v) & 0\\ 0 & 0 & 0 & (1-2v)/2\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\end{Bmatrix}$$

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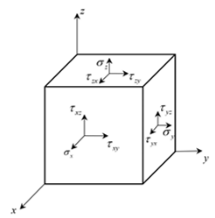
2.6 Equilibrium equations

- **Three-dimensional problems**
  - The linear momentum or equilibrium equations for the three-dimensional behavior of a solid may be written in Cartesian coordinates as



$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+b_x=0$$
$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+b_y=0$$
$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+b_z=0$$

2.6 Equilibrium equations

- **Three-dimensional problems**
  - The equilibrium equations in Cartesian coordinates may be written in a matrix Voigt form as

$$\mathbf{S}^{\mathrm{T}}\boldsymbol{\sigma}+\mathbf{b}=0$$

S is the same differential operator, **b** is the vector of body forces given as

$$\mathbf{b}=\begin{bmatrix}b_x & b_y & b_z\end{bmatrix}^{\mathrm{T}}$$

2.6 Equilibrium equations

- **Two-dimensional problems**
  - The linear momentum or equilibrium equations for the two-dimensional plane problems behavior of a solid may be written in Cartesian coordinates as

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+b_x=0$$
$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+b_y=0$$

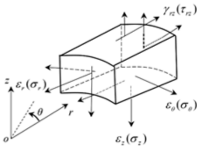
and in matrix form

$$\mathbf{S}_p^{\mathrm{T}}\boldsymbol{\sigma}+\mathbf{b}=0$$

2.6 Equilibrium equations

■ **Two-dimensional problems**

- The linear momentum or equilibrium equations for the two-dimensional axisymmetric problems behavior of a solid may be written in Cartesian coordinates as



$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \tau_{zr}}{\partial z} + b_r = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + b_z = 0$$

and the differential operator on equilibrium in matrix form

$$\overline{\mathbf{S}}_a^{\mathrm{T}} = \begin{bmatrix} \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) & 0 & -\frac{1}{r} & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} \end{bmatrix}$$

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2.7 Boundary conditions

- Displacement boundary conditions are specified at each point of the boundary Γ_u as

$$\mathbf{u} = \overline{\mathbf{u}}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_u$$

where $\bar{\mathbf{u}}$ are known values and **x** are points on the boundary.

- Traction boundary conditions are specified for each point of the boundary Γ_t and are given in terms of stresses by

$$\mathbf{t} = \mathbf{G}^{\mathrm{T}}\boldsymbol{\sigma} = \bar{\mathbf{t}}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_t$$

in which for three-dimensional problems $\mathbf{G}^{\mathrm{T}}$ is the matrix, and in two-dimensional plane problems $\mathbf{G}^{\mathrm{T}}$ reduces to $\mathbf{G}_p^{\mathrm{T}}$

$$\mathbf{G}^{\mathrm{T}} = \begin{bmatrix} n_x & 0 & 0 & n_y & 0 & n_z \\ 0 & n_y & 0 & n_x & n_z & 0 \\ 0 & 0 & n_z & 0 & n_y & n_x \end{bmatrix} \qquad \mathbf{G}_p^{\mathrm{T}} = \begin{bmatrix} n_x & 0 & 0 & n_y \\ 0 & n_y & 0 & n_x \end{bmatrix}$$

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The End

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