Computational Mechanics

(*English Course*)

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2 Weak Form of Equivalent Integration

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2 Weak Form of Equivalent Integration

■ Keywords

- **Strong form** 强形式 **Weak form** 弱形式
- **Differential** 微分 **Integration** 积分
- **Arbitrary function** 任意函数
- **Integration by parts** 分部积分
- **Derivative** 导数
- **Galerkin method** 伽辽金法
- **Shape function** 形函数
- **Stiffness** 刚度
- **Load** 载荷
- **Element location vector** 单元定位向量

3 Weak Form of Equivalent Integration

1.2 Computation procedure of finite element method

1. Define the problem to be solved by the governing differential equations. Based on the differential equations in the analysis of continuous systems, the equivalent integration form of the problem is constructed as virtual work, variational or **weak form**.
2. Select the type and order of the finite element. These elements and corresponding shape functions will be used in the analysis of discrete systems, which will be substituted into the equivalent integral form.
3. Define a set of mesh for the problem. This involves the description of the distribution of nodes and elements and the description of the boundary conditions.

3.1 Weak form of equivalent integration for differential equations

- Strong Form: governing equations (Partial Differential Equations, PDEs)
- Weak Form:
  (1) Multiply each equation by an appropriate arbitrary function.
  (2) Integrate this product over the space domain of the problem.
  (3) Reduce the order of the invoved derivatives using the integration by parts.
  (4) Introduce the boundary conditions if possible.

An arbitrary function is one that can take any value we can imagine. It can be a polynomial(多项式), a trigonometric function (三角函数), a Dirac delta function, or any other function.

3 Weak Form of Equivalent Integration

3.1 Weak form of equivalent integration for differential equations

3.2 Weak form of one-dimensional elasticity problems

3.3 Finite element computation based on weak form

3.4 Global assembly from one-dimensional elements

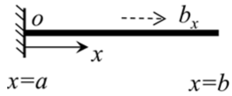
3.5 Treatments on boundary conditions

3.6 Exercises

2.8 Exercises

2.8.1 Summary the basic variables and equations of three-dimensional and two-dimensional elasticity.

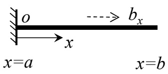
2.8.2 According to the expression methods in this chapter, provide the basic variables and equations of axial deformation of the following one-dimensional elastic cantilever beam. The coordinates of the two ends are $x=a$ and $x=b$, respectively; the axial uniform load is b_x.



- **Figure 2.5 Axial deformation of one-dimensional elastic cantilever beam.**

3.2 Weak form of one-dimensional elasticity problems

- **One-dimensional elastic cantilever beam**



- Displacemet: u strain: ε_x stress: σ_x
- Geometric equation: $\varepsilon_x = \dfrac{\partial u}{\partial x}$
- Constitutive equation: $\sigma_x = E\varepsilon_x$
- Equilibrium equation: $\dfrac{\partial \sigma_x}{\partial x} + b_x = 0$
-
- Boundary conditions:

$$u = \bar{u}\ (=0), \quad x = a$$
$$t_x = \bar{t}_x = n_x\sigma_x (=0), \quad x = b$$

Strong form

$$\frac{\partial}{\partial x}\left(E\frac{\partial u}{\partial x}\right) + b_x = 0$$

n_x is the unit outward normal vector

3.2 Weak form of one-dimensional elasticity problems

- **Weak form of equilibrium equation**

- We start by introducing **an arbitrary function $w(x)$** that is defined in the domain described by the interval $a < x < b$. Multiplying the equilibrium equation by this function we may write

$$\frac{\partial \sigma_x}{\partial x} + b_x = 0$$

$$g(\omega, u, \sigma_x) = w(x)\left(b_x + \frac{\partial \sigma_x}{\partial x}\right) = 0$$

- Integrate this product over the space domain of the problem

$$G(w, u, \sigma_x) = \int_\Omega w(x)\left(b_x + \frac{\partial \sigma_x}{\partial x}\right)\mathrm{d}x = 0$$

3.2 Weak form of one-dimensional elasticity problems

- **Weak form of equilibrium equation**

- Integrate the stress term by parts as

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

$$G(w, u, \sigma_x) = \int_\Omega w(x)\left(b_x + \frac{\partial \sigma_x}{\partial x}\right)\mathrm{d}x = 0$$

$$\Rightarrow \quad \int_\Omega w(x)\frac{\partial \sigma_x}{\partial x}dx = w(x)n_x\sigma_x\big|_\Gamma - \int_\Omega \frac{\partial w}{\partial x}\sigma_x dx$$

- Where Γ is the boundary of Ω, and n_x is the outward pointing normal to the boundary. The boundary term may be expressed in terms of the traction as

$$w(x)n_x\sigma_x\big|_\Gamma = w(x)t_x(x)\big|_\Gamma$$

where we have noted $n_x(b) = 1$ and $n_x(a) = -1$.

3.2 Weak form of one-dimensional elasticity problems

- **Weak form of equilibrium equation**

- We also again introduce the notation that u is a boundary where $u = \bar{u}$, t is a boundary where $t_x = \bar{t}_x$, and the total boundary is $\Gamma=\Gamma_u \cup \Gamma_t$. With this notation we can write the *weak form* for equilibrium as

$$G(w, u, \sigma_x) = \int_\Omega w(x)\left(b_x + \frac{\partial \sigma_x}{\partial x}\right)\mathrm{d}x = 0$$

Weak form

$$\Downarrow$$

$$G(w, u, \sigma_x) = \int_\Omega w(x)b_x\mathrm{d}x - \int_\Omega \frac{\partial w}{\partial x}\sigma_x\mathrm{d}x + wt_x\big|_{\Gamma_u} + w\bar{t}_x\big|_{\Gamma_t} = 0$$

$$\Downarrow \quad w\big|_{\Gamma_u} = 0$$

3.2 Weak form of one-dimensional elasticity problems

- **Weak form of equilibrium equation**

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Weak form

⇩

$$G(w, u, \sigma_x) = \int_\Omega w(x) b_x \mathrm{d}x - \int_\Omega \frac{\partial w}{\partial x}\sigma_x \mathrm{d}x + w\bar{t}_x\big|_{\Gamma_t} = 0$$

$\varepsilon_x = \frac{\partial u}{\partial x}$ ⇩ $\sigma_x = E\varepsilon_x$

3.2 Weak form of one-dimensional elasticity problems

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Weak form

⇩

$$G(w, u) = \int_\Omega w(x) b_x \mathrm{d}x - \int_\Omega \frac{\partial w}{\partial x} E \frac{\partial u}{\partial x}\mathrm{d}x + w\bar{t}_x\big|_{\Gamma_t} = 0$$

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