Computational Mechanics

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2	Weak Form of Equivalent Integration
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1.2 Computation procedure of finite element method

- 1. Define the problem to be solved by the **governing differential equations**. Based on the differential equations in the analysis of continuous systems, the equivalent integration form of the problem is constructed as virtual work, variational or **weak form**.
- Select the type and order of the finite element. These elements and corresponding shape functions will be used in the analysis of discrete systems, which will be substituted into the equivalent integral form.
- 3. Define a set of mesh for the problem. This involves the description of the distribution of nodes and elements and the description of the boundary conditions.

2 Weak Form of Equivalent Integration Keywords Strong form Weak form 弱形式 强形式 . Differential 微分 Integration 积分 Arbitrary function 任意函数 Integration by parts 分部积分 Derivative 导数 Galerkin method 伽辽金法 Shape function 形函数 Stiffness 刚度 Load 载荷 Element location vector 单元定位向量

3 Weak Form of Equivalent Integration

- 3.1 Weak form of equivalent integration for differential equations
- 3.2 Weak form of one-dimensional elasticity problems
- 3.3 Finite element computation based on weak form
- 3.4 Global assembly from one-dimensional elements
- 3.5 Treatments on boundary conditions
- 3.6 Exercises

3.1 Weak form of equivalent integration for differential equations

- Strong Form: governing equations (Partial Differential Equations, PDEs)
- Weak Form:
 - (1) Multiply each equation by an appropriate arbitrary function.
 - (2) Integrate this product over the space domain of the problem.(3) Use integration by parts to reduce the order of derivatives to
 - a minimum. (4) Introduce boundary conditions if possible.

An arbitrary function is one that can take any value we can imagine. It can be a polynomial(多项式), a trigonometric function (三角函数), a Dirac delta function, or any other function.

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2.8 Exercises

2.8.1 Summary the basic variables and equations of threedimensional and two-dimensional elasticity.

2.8.2 According to the expression methods in this chapter, provide the basic variables and equations of axial deformation of the following one-dimensional elastic cantilever beam. The coordinates of the two ends are x=a and x=b, respectively; the axial uniform load is b_x .

$$\underbrace{\stackrel{o}{\longrightarrow}_{x}}_{x=a} \xrightarrow{x=b_{x}}$$

• Figure 2.5 Axial deformation of one-dimensional elastic cantilever beam.

3.2 Weak form of one-dimensional elasticity problems					
• One-dimensional elastic cantilever beam $\downarrow_{x}^{o} \longrightarrow b_{x}$					
• Displacemet: <i>u</i>	strain: ε_x	x= stress:	$\sigma_x = b$		
• Geometric equation:	$\mathcal{E}_x = \frac{\partial u}{\partial x}$				
Constitutive equation	n: $\sigma_x = E\varepsilon_x$		Strong form		
• Equilibrium equation	n: $\frac{\partial \sigma_x}{\partial x} + b_x =$	= 0	$\frac{\partial}{\partial x} \left(E \frac{\partial u}{\partial x} \right) + b_x = 0$		
Boundary conditions	S: $u = \overline{u} (=0),$ $t_x = \overline{t}_x = n_x \sigma_x (=0),$	x = a $x = b$	n_x is the unit outward normal vector 9		

3.2 Weak form of one-dimensional elasticity problems

Weak form of equilibrium equation

 We start by introducing an arbitrary function w(x) that is defined in the domain described by the interval a < x < b. Multiplying the equilibrium equation by this function we may write

$$g(\omega, u, \sigma_x) = \boxed{w(x)\left(b_x + \frac{\partial \sigma_x}{\partial x}\right)} = 0$$

• Integrate this product over the space domain of the problem

$$G(w, u, \sigma_x) = \int_{\Omega} w(x) \left(b_x + \frac{\partial \sigma_x}{\partial x} \right) dx = 0$$







■ Weak form of equilibrium equation

• We also again introduce the notation that u is a boundary where $u = \bar{u}$, t is a boundary where $t_x = \bar{t}_x$, and the total boundary is $\Gamma = \Gamma_u \cup \Gamma_r$. With this notation we can write the *weak form* for equilibrium as $\left(-\frac{1}{2} \overline{\sigma_r} \right)$

$$G(w, u, \sigma_{x}) = \int_{\Omega} w(x) \left[b_{x} + \left[\frac{c \sigma_{x}}{2 \alpha} \right] \right] dx = 0$$

$$Weak form$$

$$\bigcup$$

$$G(w, u, \sigma_{x}) = \int_{\Omega} w(x) b_{x} dx \left[- \int_{\Omega} \frac{\partial w}{\partial x} \sigma_{x} dx + wt_{x} \Big|_{\Gamma_{x}} + w\bar{t}_{x} \Big|_{\Gamma_{x}} \right] = 0$$

$$\bigcup$$

$$W_{r_{x}} = 0$$

$$I_{12}$$

3.2 Weak form of one-dimensional elasticity problems
Weak form of equilibrium equation
• We also again introduce the notation that *u* is a boundary where
$$u = \bar{u}$$
,
t is a boundary where $t_x = \bar{t}_x$, and the total boundary is $\Gamma = \Gamma_u \cup \Gamma_r$.
With this notation we can write the *weak form* for equilibrium as
 $G(w, u, \sigma_x) = \int_{\Omega} w(x) \left(b_x + \frac{1}{2} \frac{\partial \sigma_x}{\partial x} \right) dx = 0$
Weak form
 $G(w, u, \sigma_x) = \int_{\Omega} w(x) b_x dx - \int_{\Omega} \frac{\partial w}{\partial x} \sigma_x dx + w\bar{t}_x |_{\Gamma_r} = 0$
 $\overline{v_x} = \frac{\partial u}{\partial x} \int_{\Omega} \overline{v_x} (\sigma_x = E\varepsilon_x)$
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3.3 Finite element computation based on weak form

Galerkin method

• To construct an approximate solution we express the displacement u(x) in terms of a set of specified functions multiplied by unknown parameters.

$$u(x) \approx \hat{u}(x) = \sum_{n=1}^{N} \phi_n(x) a_n + u_{\overline{z}}(x)$$

• In a similar way, we write the arbitrary function *w*(*x*) in terms of an equal number of specified functions and arbitrary parameters. These may be expressed as

$$w(x) \approx \hat{w}(x) = \sum_{m=1}^{N} \psi_m(x) w_m$$

In the above form we assume that both $\phi_n(x)$ and $\psi_m(x)$ are zero at all locations where the boundary **DISPLACEMENT** is specified.

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The function u_s(x) is then specified as any function that satisfies the DISPLACEMENT boundary condition. For example, if the displacement must satisfy u(L) = d
 on the domain 0 < x < L, this function may be taken as

$$u_{\overline{b}}(x) = \frac{x}{L}\overline{d} = \phi_{\overline{b}}(x)\overline{d}$$

























The End