Computational Mechanics

(*English Course*)

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3 Weak Form of Equivalent Integration

3.3 Finite element computation based on weak form

■ **Galerkin method**

- To construct an approximate solution we express the displacement $u(x)$ in terms of a set of specified functions multiplied by unknown parameters.

$$u(x) \approx \hat{u}(x) = \sum_{n=1}^{N} \phi_n(x) a_n + u_{\bar{\delta}}(x)$$

- In a similar way, we write the arbitrary function $w(x)$ in terms of an equal number of specified functions and arbitrary parameters. These may be expressed as

$$w(x) \approx \hat{w}(x) = \sum_{m=1}^{N} \psi_m(x) w_m$$

In the above form we assume that both $\phi_n(x)$ and $\psi_m(x)$ are zero at all locations where the boundary **DISPLACEMENT** is specified.

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3.3 Finite element computation based on weak form

- The function $u_{\bar{\delta}}(x)$ is then specified as any function that satisfies the DISPLACEMENT boundary condition. For example, if the displacement must satisfy $u(L) = \bar{d}$ on the domain $0 < x < L$, this function may be taken as

$$u_{\bar{\delta}}(x) = \frac{x}{L} \bar{d} = \phi_{\bar{\delta}}(x) \bar{d}$$

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3.3 Finite element computation based on weak form

■ **Galerkin method**

$$G(w, u) = \int_{\Omega} w(x) b_x \mathrm{d}x - \int_{\Omega} \frac{\partial w}{\partial x} E \frac{\partial u}{\partial x} \mathrm{d}x + w \bar{t}_x \Big|_{\Gamma_t} = 0$$

$$u(x) \approx \hat{u}(x) = \sum_{n=1}^{N} \phi_n(x) a_n + u_{\bar{\delta}}(x) \qquad u_{\bar{\delta}}(x) = \phi_{\bar{\delta}}(x) \bar{d}$$

$$w(x) \approx \hat{w}(x) = \sum_{m=1}^{N} \psi_m(x) w_m$$

- An approximate Galerkin solution for the elasticity problem

$$\hat{G}(\hat{w}, \hat{u}) = \sum_{m=1}^{N} w_m \int_{\Omega} \psi_m b_x \,\mathrm{d}x - \sum_{m=1}^{N} w_m \int_{\Omega} \frac{\mathrm{d}\psi_m}{\mathrm{d}x} E \left[\sum_{n=1}^{N} \frac{d\phi_n}{dx} a_n + \frac{d\phi_{\bar{\delta}}}{dx} \bar{d} \right] \mathrm{d}x + \sum_{m=1}^{N} w_m \psi_m(x) \bar{t}_x \Big|_{\Gamma_t} = 0$$

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3.3 Finite element computation based on weak form

$$\hat{G}(\hat{w}, \hat{u}) = \sum_{m=1}^{N} w_m \int_{\Omega} \psi_m b_x \,\mathrm{d}x - \sum_{m=1}^{N} w_m \int_{\Omega} \frac{\mathrm{d}\psi_m}{\mathrm{d}x} E \left[\sum_{n=1}^{N} \frac{d\phi_n}{dx} a_n + \frac{d\phi_{\bar{\delta}}}{dx} \bar{d} \right] \mathrm{d}x + \sum_{m=1}^{N} w_m \psi_m(x) \bar{t}_x \Big|_{\Gamma_t} = 0$$

$$f_m = \int_{\Omega} \psi_m b_x \,\mathrm{d}x - \int_{\Omega} \frac{\mathrm{d}\psi_m}{\mathrm{d}x} E \frac{\mathrm{d}\phi_{\bar{\delta}}}{\mathrm{d}x} \bar{d} \,\mathrm{d}x + \psi_m \bar{t}_x \Big|_{\Gamma_t}$$

load matrix

$$K_{mn} = \int_{\Omega} \frac{\mathrm{d}\psi_m}{\mathrm{d}x} E \frac{\mathrm{d}\phi_n}{\mathrm{d}x} \mathrm{d}x$$

stiffness matrix

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3.3 Finite element computation based on weak form

$$\hat{G}(\hat{w},\hat{u})=\sum_{m=1}^{N} w_m \int_\Omega \psi_m b_x \, dx-\sum_{m=1}^{N} w_m \int_\Omega \frac{d\psi_m}{dx} E\left[\sum_{n=1}^{N}\frac{d\phi_n}{dx}a_n+\frac{d\phi_{\bar{b}}}{dx}\bar{d}\right]dx + \sum_{m=1}^{N} w_m \psi_m(x)\bar{t}_x\Big|_{\Gamma_t}=0$$

$$f_m=\int_\Omega \psi_m b_x \, dx-\int_\Omega \frac{d\psi_m}{dx}E\frac{d\phi_{\bar{b}}}{dx}\bar{d}\,dx+\psi_m \bar{t}_x\Big|_{\Gamma_t}$$

load matrix

$$K_{mn}=\int_\Omega \frac{d\psi_m}{dx}E\frac{d\phi_n}{dx}\,dx$$

stiffness matrix

$$\sum_{m=1}^{N} w_m\left(-\sum_{n=1}^{N}K_{mn}a_n+f_m\right)=0 \quad \Longrightarrow \quad \sum_{m=1}^{N} w_m\left(\sum_{n=1}^{N}K_{mn}a_n-f_m\right)=0$$

3.3 Finite element computation based on weak form

- Since the parameters w_m are arbitrary, the expression multiplying each one must be zero. This leads to the set of equations:

$$\sum_{m=1}^{N} w_m\left(\sum_{n=1}^{N}K_{mn}a_n-f_m\right)=0$$

$$\Downarrow$$

$$\sum_{n=1}^{N}K_{mn}a_n=f_m, \quad m=1,2,\ldots,N$$

- The original problem of partial differential equations has been reduced to a set of algebraic equations.

3.3 Finite element computation based on weak form

- Matrix form of stiffness equation

$$\sum_{n=1}^{N}K_{mn}a_n=f_m, \quad m=1,2,\ldots,N$$

$$\mathbf{Ka}=\mathbf{f}$$

$$\mathbf{K}=\begin{bmatrix}K_{11} & K_{12} & \cdots & K_{1N}\\ K_{21} & K_{22} & & K_{1N}\\ \vdots & & & \vdots\\ K_{N1} & K_{N2} & \cdots & K_{NN}\end{bmatrix} \quad \text{and} \quad \mathbf{f}=\begin{Bmatrix}f_1\\ f_2\\ \vdots\\ f_N\end{Bmatrix} \quad \mathbf{a}=\begin{Bmatrix}a_1\\ a_2\\ \vdots\\ a_N\end{Bmatrix}$$

- Formal solution

$$\mathbf{a}=\mathbf{K}^{-1}\mathbf{f}$$

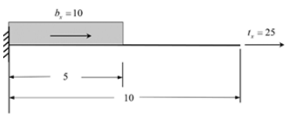
- Displacement and stress solution

$$u(x)\approx\hat{u}(x)=\sum_{n=1}^{N}\phi_n(x)a_n+u_{\bar{b}}(x) \qquad \hat{\sigma}_x(x)=E\frac{\partial\hat{u}(x)}{\partial x}=E\left(\sum_{n=1}^{N}\frac{d\phi_n}{dx}a_n+\frac{d\phi_{\bar{b}}(x)}{dx}\bar{d}\right)$$

3.3 Finite element computation based on weak form

- **Example 3.1. Solutions of Galerkin method for one-dimensional elastic cantilever beam**

- As an example we consider a static problem with length 10 units and E = 1000. Figure 3.1 shows the problem to solve.



- **Figure 3.1 One-dimensional elastic cantilever beam in Example 3.1.**

- Loading $b_x=\begin{cases}10 & \text{for } 0<x<5\\ 0 & \text{for } 5<x<10\end{cases}$

- Boundary conditions

$$u(0)=0 \quad \text{and} \quad \bar{t}_x(10)=25$$

3.3 Finite element computation based on weak form

- Weak form

$$G(w,u)=\int_\Omega w(x)b_x dx-\int_\Omega \frac{\partial w}{\partial x}E\frac{\partial u}{\partial x}dx+w\bar{t}_x\Big|_{\Gamma_t}=0$$

$$\int_0^{10}\frac{\partial w}{\partial x}1000\frac{\partial u}{\partial x}dx-\int_0^5 w(x)10\,dx-w(10)25=0$$

- consider an approximate solution

$$u(x)\approx\hat{u}(x)=\sum_{n=1}^{N}\phi_n(x)a_n+u_{\bar{b}}(x) \qquad w(x)\approx\hat{w}(x)=\sum_{m=1}^{N}\psi_m(x)w_m$$

$$\hat{u}(x)=\sum_{n=1}^{N}\left(\frac{x}{10}\right)^n a_n \quad \text{and} \quad \hat{w}(x)=\sum_{m=1}^{N}\left(\frac{x}{10}\right)^m w_m$$

3.3 Finite element computation based on weak form

- Weak form

$$\int_0^{10}\frac{\partial w}{\partial x}1000\frac{\partial u}{\partial x}dx-\int_0^5 w(x)10\,dx-w(10)25=0$$

$$\hat{u}(x)=\sum_{n=1}^{N}\left(\frac{x}{10}\right)^n a_n \quad \text{and} \quad \hat{w}(x)=\sum_{m=1}^{N}\left(\frac{x}{10}\right)^m w_m$$

$$\Downarrow$$

$$\sum_{n=1}^{N}\left[\int_0^{10}10mn\left(\frac{x}{10}\right)^{m-1}\left(\frac{x}{10}\right)^{n-1}dx\right]a_n=\int_0^5\left(\frac{x}{10}\right)^m 10\,dx+25$$

3.3 Finite element computation based on weak form

- Weak form

$$\sum_{m=1}^{N}\left[\int_0^{10}10mn\left(\frac{x}{10}\right)^{m-1}\left(\frac{x}{10}\right)^{n-1}\mathrm{d}x\right]a_n=\int_0^5\left(\frac{x}{10}\right)^m 10\,\mathrm{d}x+25$$

$$\downarrow$$

$$K_{mn}=\int_0^{10}10mn\left(\frac{x}{10}\right)^{m+n-2}\mathrm{d}x=\frac{100mn}{m+n-1}$$

$$f_m=\int_0^5\left(\frac{x}{10}\right)^m 10\,\mathrm{d}x+25=\frac{100}{m+1}\left(\frac{1}{2}\right)^{m+1}+25$$

3.3 Finite element computation based on weak form

- For example, *m*=*n*=2, the stiffness matrix and load matrix are given by

$$\mathbf{K}=\begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}=\begin{bmatrix}100 & 100\\ 100 & 400/3\end{bmatrix}$$

$$\mathbf{f}=\begin{Bmatrix}f_1\\ f_2\end{Bmatrix}=\begin{Bmatrix}75/2\\ 175/6\end{Bmatrix}$$

- Matrix form of stiffness equation

$$\mathbf{Ka}=\mathbf{f}$$

3.5 Treatments on boundary conditions

- Formal solution $\mathbf{a}=\mathbf{K}^{-1}\mathbf{f}$

$$\begin{bmatrix}100 & 100\\ 100 & 400/3\end{bmatrix}\begin{Bmatrix}a_1\\ a_2\end{Bmatrix}=\begin{Bmatrix}75/2\\ 175/6\end{Bmatrix}\Rightarrow\mathbf{a}=\begin{Bmatrix}a_1\\ a_2\end{Bmatrix}=\begin{Bmatrix}0.62500\\ -0.25000\end{Bmatrix}$$

- Displacement and stress solution

$$u(x)\approx\hat{u}(x)=\sum_{n=1}^{N}\phi_n(x)a_n+u_5(x)\qquad \hat{\sigma}_x(x)=E\frac{\partial\hat{u}(x)}{\partial x}=E\left(\sum_{n=1}^{N}\frac{\mathrm{d}\phi_n}{\mathrm{d}x}a_n+\frac{\mathrm{d}\phi_5(x)}{\mathrm{d}x}\bar{d}\right)$$

$$u(x)\approx\hat{u}(x)=\sum_{n=1}^{2}\left(\frac{x}{10}\right)^n a_n=\frac{x}{10}\times0.625+\left(\frac{x}{10}\right)^2\times(-0.25)=-0.0025x^2+0.0625x$$

$$\hat{\sigma}_x(x)=E\frac{\partial\hat{u}(x)}{\partial x}=E\left(\sum_{n=1}^{2}\frac{\mathrm{d}\phi_n}{\mathrm{d}x}a_n\right)=-5x+62.5$$

3.3 Finite element computation based on weak form

- Similarly, we get the solutions under different number of terms

N-terms	a_1	a_2	a_3	a_4	a_5
1	0.37500				
2	0.62500	-0.25000			
3	0.78125	-0.71875	0.31250		
4	0. 78125	-0.71875	0.31250	0.00000	
5	0.73437	-0.25000	-1.09275	1.64063	0.65625

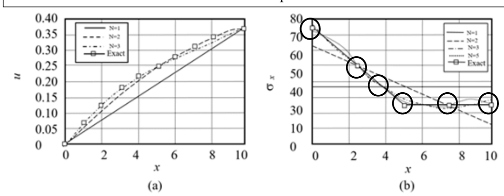
- **Table 3.1 Parameters for one-dimensional elastic cantilever beam in Example 3.1.**

- Discard terms that do not contribute to the solution.
- Does not lead to convergent behavior.

3.3 Finite element computation based on weak form

- Displacement and stress solution

1. The displacement at the free end is the same no matter how many terms we use. This often happens in one-dimensional static problems but, unfortunately, is seldom true in higher dimensional problems.
2. The solution for stress converges more slowly than that for the displacements;
3. however, once again we observe that some points are more accurate than others. These we shall call ***super-convergent* points** and these points play an important role in our later discussion on error estimates and adaptive refinement of solutions.



- **Figure 3.2 Displacement and stress solutions in Example 3.1 based on Galerkin method using *N*-terms solutions: (a) displacement and (b) stress.**

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