3 Weak Form of Equivalent Integration

3.3 Finite element computation based on weak form

■ **Finite element computation**

- A more convenient method to construct the approximating functions ϕ_n and ψ_m are obtained by dividing the domain to be analyzed into small regular shaped regions. For example, we can divide the one-dimensional region between a and b into a set of "M" small finite segments by defining a set of N points x_i such that

$$x_1 = a, \quad x_i < x_{i+1} \quad \text{and} \quad x_N = b$$

For a one-dimensional problem we can let each increment define a *finite element* domain (or more simply, an *element*) and the set of points define the *nodes* (*finite element mesh* or *mesh*).

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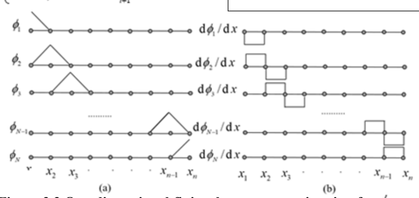
3.3 Finite element computation based on weak form

- A simple set of continuous polynomial approximating functions

$$\phi_i = \begin{cases} 0, & x < x_{i-1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i < x \le x_{i+1} \\ 0, & x > x_{i+1} \end{cases}$$

C_0 function since only the function is continuous in x, whereas the first derivative is only piecewise continuous with the discontinuities located at the nodes.



- **Figure 3.3 One-dimensional finite element approximation for ϕ_i : (a) functions and (b) derivatives.**

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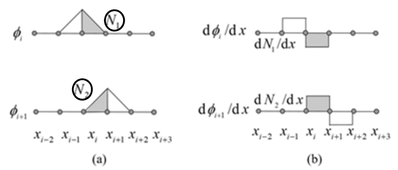
3.3 Finite element computation based on weak form

- Integrals over each element in the weak form

$$\int_\Omega (\cdot)\,\mathrm{d}x = \sum_{i=1}^{M} \int_{x_i}^{x_{i+1}} (\cdot)\,\mathrm{d}x \equiv \sum_e \int_{\Omega_e} (\cdot)\,\mathrm{d}x$$

Considering any interval $[x_i, x_{i+1}]$, we note that each interval is defined by the same two local functions N_1 and N_2. We call these the *shape functions* (形函数) for the element.



- **Figure 3.4 One-dimensional finite element shape functions: (a) functions and (b) derivatives.**

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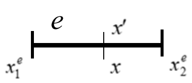
3.3 Finite element computation based on weak form

- Define local nodal coordinates x' on each element with x_1^e and x_2^e.



$$x' = x - x_1^e \qquad h_e = x_2^e - x_1^e$$

- Local coordinate system

$$N_1(x') = 1 - \frac{x'}{h_e} \quad \text{and} \quad N_2(x') = \frac{x'}{h_e}$$

$$\frac{\mathrm{d}N_1}{\mathrm{d}x} = \frac{\mathrm{d}N_1}{\mathrm{d}x'} = -\frac{1}{h_e} \quad \text{and} \quad \frac{\mathrm{d}N_2}{\mathrm{d}x} = \frac{\mathrm{d}N_2}{\mathrm{d}x'} = \frac{1}{h_e}$$

$$\phi_i = \begin{cases} 0, & x < x_{i-1} \\ N_2(x):\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \le x \le x_i \\ N_1(x):\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i < x \le x_{i+1} \\ 0, & x > x_{i+1} \end{cases}$$

- Displacements and arbitrary function on the element

$$\hat{u}^e = N_1(x')\,\hat{u}_1^e + N_2(x')\,\hat{u}_2^e$$
$$\hat{w}^e = N_1(x')\,\hat{w}_1^e + N_2(x')\,\hat{w}_2^e$$

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3.3 Finite element computation based on weak form

- Weak form in global domain

$$G(w,u) = \int_\Omega w(x) b_x \mathrm{d}x - \int_\Omega \frac{\partial w}{\partial x} E \frac{\partial u}{\partial x} \mathrm{d}x + w\bar{t}_x\big|_{\Gamma_t} = 0$$

- Weak form in each element

$$\hat{G}(\hat{w},\hat{u}) = \hat{G}_\sigma(\hat{w},\hat{u}) - \hat{G}_f(\hat{w},\hat{u}) - \hat{w}(x)\bar{t}_x\big|_{\Gamma_t}$$

$$\hat{G}_\sigma(\hat{w},\hat{u}) = \sum_{e=1}^{M} \begin{bmatrix} \hat{w}_1^e & \hat{w}_2^e \end{bmatrix} \int_0^{h_e} \begin{Bmatrix} \dfrac{\mathrm{d}N_1}{\mathrm{d}x'} \\ \dfrac{\mathrm{d}N_2}{\mathrm{d}x'} \end{Bmatrix} E_e \begin{bmatrix} \dfrac{\mathrm{d}N_1}{\mathrm{d}x'} & \dfrac{\mathrm{d}N_2}{\mathrm{d}x'} \end{bmatrix} dx' \begin{Bmatrix} \hat{u}_1^e \\ \hat{u}_2^e \end{Bmatrix}$$

$$\hat{G}_f(\hat{w},\hat{u}) = \sum_{e=1}^{M} \begin{bmatrix} \hat{w}_1^e & \hat{w}_2^e \end{bmatrix} \int_0^{h_e} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} b_x\, \mathrm{d}x'$$

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3.3 Finite element computation based on weak form

- Each element can be evaluated as element stiffness matrix and load matrix

$$\mathbf{K}^e = \int_0^{h_e} \begin{Bmatrix} \dfrac{\mathrm{d}\,N_1}{\mathrm{d}\,x'} \\ \dfrac{\mathrm{d}\,N_2}{\mathrm{d}\,x'} \end{Bmatrix} E \begin{bmatrix} \dfrac{\mathrm{d}\,N_1}{\mathrm{d}\,x'} & \dfrac{\mathrm{d}\,N_2}{\mathrm{d}\,x'} \end{bmatrix} dx' = \begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix}$$

$$\mathbf{f}^e = \int_0^{h_e} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} b_x \,\mathrm{d}\,x' = \begin{Bmatrix} f_1^e \\ f_2^e \end{Bmatrix}$$

- Element stiffness matrix and load matrix

E_e and b_x are constant $\quad N_1(x') = 1 - \dfrac{x'}{h_e}$ and $N_2(x') = \dfrac{x'}{h_e}$

$$\mathbf{K}^e = \frac{E_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{f}^e = \frac{1}{2} b_x h_e \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

3.3 Finite element computation based on weak form

- Weak form in global domain

$$G(w,\,u) = \int_\Omega w(x) b_x \mathrm{d}x - \int_\Omega \frac{\partial w}{\partial x} E \frac{\partial u}{\partial x} \mathrm{d}x + w\bar{t}_x\big|_{\Gamma_t} = 0$$

- Weak form in each element

$$\hat{G}(\hat{w},\hat{u}) = \hat{G}_\sigma(\hat{w},\hat{u}) - \hat{G}_f(\hat{w},\hat{u}) - \hat{w}(x)\bar{t}_x\big|_{\Gamma_t} = 0$$

$$\hat{G}_\sigma(\hat{w},\hat{u}) = \sum_{e=1}^{M} \begin{bmatrix} \hat{w}_1^e & \hat{w}_2^e \end{bmatrix} \int_0^{h_e} \begin{Bmatrix} \dfrac{\mathrm{d}\,N_1}{\mathrm{d}\,x'} \\ \dfrac{\mathrm{d}\,N_2}{\mathrm{d}\,x'} \end{Bmatrix} E_e \begin{bmatrix} \dfrac{\mathrm{d}\,N_1}{\mathrm{d}\,x'} & \dfrac{\mathrm{d}\,N_2}{\mathrm{d}\,x'} \end{bmatrix} \mathrm{d}\,x' \begin{Bmatrix} \hat{u}_1^e \\ \hat{u}_2^e \end{Bmatrix} = \sum_{e=1}^{M} \begin{bmatrix} \hat{w}_1^e & \hat{w}_2^e \end{bmatrix} \mathbf{K}^e \begin{Bmatrix} \hat{u}_1^e \\ \hat{u}_2^e \end{Bmatrix}$$

$$\hat{G}_f(\hat{w},\hat{u}) = \sum_{e=1}^{M} \begin{bmatrix} \hat{w}_1^e & \hat{w}_2^e \end{bmatrix} \int_0^{h_e} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} b_x \,\mathrm{d}\,x' = \sum_{e=1}^{M} \begin{bmatrix} \hat{w}_1^e & \hat{w}_2^e \end{bmatrix} \mathbf{f}^e \,\mathrm{d}\,x'$$

3.3 Finite element computation based on weak form

- Weak form in global domain

$$G(w,\,u) = \int_\Omega w(x) b_x \mathrm{d}x - \int_\Omega \frac{\partial w}{\partial x} E \frac{\partial u}{\partial x} \mathrm{d}x + w\bar{t}_x\big|_{\Gamma_t} = 0$$

- Weak form in each element

$$\hat{G}(\hat{w},\hat{u}) = \hat{G}_\sigma(\hat{w},\hat{u}) - \hat{G}_f(\hat{w},\hat{u}) - \hat{w}(x)\bar{t}_x\big|_{\Gamma_t} = 0$$

- Since $\hat{w}_2^e = \hat{w}_1^{e+1}$

$$\hat{w}_1^1 f_1\,(\mathbf{K}^e, \mathbf{f}^e, \mathbf{u}^e) + \hat{w}_1^2 f_2\,(\mathbf{K}^e, \mathbf{f}^e, \mathbf{u}^e) + \ldots + \hat{w}_1^{N-1} f_{N-1}(\mathbf{K}^e, \mathbf{f}^e, \mathbf{u}^e) + \hat{w}_2^{N-1} f_N(\mathbf{K}^e, \mathbf{f}^e, \mathbf{u}^e) = 0$$

3.3 Finite element computation based on weak form

- Weak form in global domain

$$G(w,\,u) = \int_\Omega w(x) b_x \mathrm{d}x - \int_\Omega \frac{\partial w}{\partial x} E \frac{\partial u}{\partial x} \mathrm{d}x + w\bar{t}_x\big|_{\Gamma_t} = 0$$

- Weak form in each element

$$\hat{G}(\hat{w},\hat{u}) = \hat{G}_\sigma(\hat{w},\hat{u}) - \hat{G}_f(\hat{w},\hat{u}) - \hat{w}(x)\bar{t}_x\big|_{\Gamma_t} = 0$$

- Since the parameters $\hat{w}_i^e$ are arbitrary, the expression multiplying each one must be zero. This leads to the set of equations:

$$\Rightarrow \begin{cases} f_1\,(\mathbf{K}^e, \mathbf{f}^e, \mathbf{u}^e) = 0 \\ f_2\,(\mathbf{K}^e, \mathbf{f}^e, \mathbf{u}^e) = 0 \\ \quad\vdots \\ f_{N-1}(\mathbf{K}^e, \mathbf{f}^e, \mathbf{u}^e) = 0 \\ f_N\,(\mathbf{K}^e, \mathbf{f}^e, \mathbf{u}^e) = 0 \end{cases} \Rightarrow \mathbf{Ku} - \mathbf{f} = \mathbf{0} \;\Rightarrow \mathbf{Ku} = \mathbf{f}$$

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