Computational Mechanics

(*English Course*)

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3 Weak Form of Equivalent Integration

3.3 Finite element computation based on weak form

■ **Finite element computation**

- A more convenient method to construct the approximating functions ϕ_n and ψ_m are obtained by dividing the domain to be analyzed into small regular shaped regions. For example, we can divide the one-dimensional region between a and b into a set of "M" finite small segments by defining a set of N points x_i such that

$$x_1 = a, \quad x_i < x_{i+1} \quad \text{and} \quad x_N = b$$

For a one-dimensional problem we can let each increment define a *finite element* domain (or more simply, an *element*) and the set of points define the *nodes* (*finite element mesh* or *mesh*).

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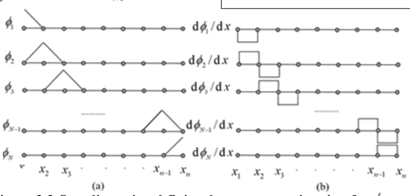
3.3 Finite element computation based on weak form

- A simple set of continuous polynomial approximating functions

$$\phi_i = \begin{cases} 0, & x < x_{i-1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i < x \le x_{i+1} \\ 0, & x > x_{i+1} \end{cases}$$

C_0 function since only the function is continuous in x, whereas the first derivative is only piecewise continuous with the discontinuities located at the nodes.



- **Figure 3.3 One-dimensional finite element approximation for ϕ_i : (a) functions and (b) derivatives.**

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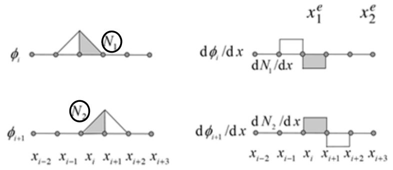
3.3 Finite element computation based on weak form

- Integrals over each element in the weak form

$$\int_\Omega (\cdot)\,\mathrm{d}x = \sum_{I=1}^{M} \int_{x_I}^{x_{I+1}} (\cdot)\,\mathrm{d}x \equiv \sum_e \int_{\Omega_e} (\cdot)\,\mathrm{d}x$$

Considering any interval $[x_i, x_{i+1}]$, we note that each interval is defined by the same two local functions N_1 and N_2. We call these the *shape functions* (形函数) for the element.



- **Figure 3.4 One-dimensional finite element shape functions: (a) functions and (b) derivatives.**

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3.3 Finite element computation based on weak form

- Define local nodal coordinates on each element as x_1^e and x_2^e.

$$x' = x - x_1^e \qquad h_e = x_2^e - x_1^e$$

- Local coordinate system

$$N_1(x') = 1 - \frac{x'}{h_e} \quad \text{and} \quad N_2(x') = \frac{x'}{h_e}$$

$$\frac{\mathrm{d}N_1}{\mathrm{d}x} = \frac{\mathrm{d}N_1}{\mathrm{d}x'} = -\frac{1}{h_e} \quad \text{and} \quad \frac{\mathrm{d}N_2}{\mathrm{d}x} = \frac{\mathrm{d}N_2}{\mathrm{d}x'} = \frac{1}{h_e}$$

- Displacements and arbitrary function on the element

$$\hat{u}^e = N_1(x')\,\hat{u}_1^e + N_2(x')\,\hat{u}_2^e$$
$$\hat{w}^e = N_1(x')\,\hat{w}_1^e + N_2(x')\,\hat{w}_2^e$$

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3.3 Finite element computation based on weak form

- Weak form in global domain

$$G(w,u)=\int_\Omega w(x)b_x\,\mathrm{d}x-\int_\Omega \frac{\partial w}{\partial x}E\frac{\partial u}{\partial x}\mathrm{d}x+w\bar{t}_x\Big|_{\Gamma_t}=0$$

- Weak form in each element

$$\hat{G}(\hat{w},\hat{u})=\hat{G}_\sigma(\hat{w},\hat{u})-\hat{G}_f(\hat{w},\hat{u})-\hat{w}(x)\bar{t}_x\Big|_{\Gamma_t}=0$$

$$\hat{G}_\sigma(\hat{w},\hat{u})=\sum_{e=1}^{M}\begin{bmatrix}\hat{w}_1^e & \hat{w}_2^e\end{bmatrix}\int_0^{h_e}\begin{Bmatrix}\dfrac{\mathrm{d}N_1}{\mathrm{d}x'}\\ \dfrac{\mathrm{d}N_2}{\mathrm{d}x'}\end{Bmatrix}E_e\begin{bmatrix}\dfrac{\mathrm{d}N_1}{\mathrm{d}x'} & \dfrac{\mathrm{d}N_2}{\mathrm{d}x'}\end{bmatrix}\mathrm{d}x'\begin{Bmatrix}\hat{u}_1^e\\ \hat{u}_2^e\end{Bmatrix}$$

$$\hat{G}_f(\hat{w},\hat{u})=\sum_{e=1}^{M}\begin{bmatrix}\hat{w}_1^e & \hat{w}_2^e\end{bmatrix}\int_0^{h_e}\begin{Bmatrix}N_1\\ N_2\end{Bmatrix}b_x\,\mathrm{d}x'$$

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3.3 Finite element computation based on weak form

- Each element can be evaluated as element stiffness matrix and load matrix

$$\mathbf{K}^e=\int_0^{h_e}\begin{Bmatrix}\dfrac{\mathrm{d}N_1}{\mathrm{d}x'}\\ \dfrac{\mathrm{d}N_2}{\mathrm{d}x'}\end{Bmatrix}E\begin{bmatrix}\dfrac{\mathrm{d}N_1}{\mathrm{d}x'} & \dfrac{\mathrm{d}N_2}{\mathrm{d}x'}\end{bmatrix}\mathrm{d}x'=\begin{bmatrix}K_{11}^e & K_{12}^e\\ K_{21}^e & K_{22}^e\end{bmatrix}$$

$$\mathbf{f}^e=\int_0^{h_e}\begin{Bmatrix}N_1\\ N_2\end{Bmatrix}b_x\,\mathrm{d}x'=\begin{Bmatrix}f_1^e\\ f_2^e\end{Bmatrix}$$

- Element stiffness matrix and load matrix

E_e and b_x are constant $\quad N_1(x')=1-\dfrac{x'}{h_e}$ and $N_2(x')=\dfrac{x'}{h_e}$

$$\mathbf{K}^e=\frac{E_e}{h_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\quad\text{and}\quad\mathbf{f}^e=\frac{1}{2}b_x h_e\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

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3.3 Finite element computation based on weak form

- Weak form in global domain

$$G(w,u)=\int_\Omega w(x)b_x\,\mathrm{d}x-\int_\Omega \frac{\partial w}{\partial x}E\frac{\partial u}{\partial x}\mathrm{d}x+w\bar{t}_x\Big|_{\Gamma_t}=0$$

- Weak form in each element

$$\hat{G}(\hat{w},\hat{u})=\hat{G}_\sigma(\hat{w},\hat{u})-\hat{G}_f(\hat{w},\hat{u})-\hat{w}(x)\bar{t}_x\Big|_{\Gamma_t}=0$$

$$\hat{G}_\sigma(\hat{w},\hat{u})=\sum_{e=1}^{M}\begin{bmatrix}\hat{w}_1^e & \hat{w}_2^e\end{bmatrix}\int_0^{h_e}\begin{Bmatrix}\dfrac{\mathrm{d}N_1}{\mathrm{d}x'}\\ \dfrac{\mathrm{d}N_2}{\mathrm{d}x'}\end{Bmatrix}E_e\begin{bmatrix}\dfrac{\mathrm{d}N_1}{\mathrm{d}x'} & \dfrac{\mathrm{d}N_2}{\mathrm{d}x'}\end{bmatrix}\mathrm{d}x'\begin{Bmatrix}\hat{u}_1^e\\ \hat{u}_2^e\end{Bmatrix}=\sum_{e=1}^{M}\begin{bmatrix}\hat{w}_1^e & \hat{w}_2^e\end{bmatrix}\mathbf{K}^e\begin{Bmatrix}\hat{u}_1^e\\ \hat{u}_2^e\end{Bmatrix}$$

$$\hat{G}_f(\hat{w},\hat{u})=\sum_{e=1}^{M}\begin{bmatrix}\hat{w}_1^e & \hat{w}_2^e\end{bmatrix}\int_0^{h_e}\begin{Bmatrix}N_1\\ N_2\end{Bmatrix}b_x\,\mathrm{d}x'=\sum_{e=1}^{M}\begin{bmatrix}\hat{w}_1^e & \hat{w}_2^e\end{bmatrix}\mathbf{f}^e\,\mathrm{d}x'$$

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3.3 Finite element computation based on weak form

- Weak form in global domain

$$G(w,u)=\int_\Omega w(x)b_x\,\mathrm{d}x-\int_\Omega \frac{\partial w}{\partial x}E\frac{\partial u}{\partial x}\mathrm{d}x+w\bar{t}_x\Big|_{\Gamma_t}=0$$

- Weak form in each element

$$\hat{G}(\hat{w},\hat{u})=\hat{G}_\sigma(\hat{w},\hat{u})-\hat{G}_f(\hat{w},\hat{u})-\hat{w}(x)\bar{t}_x\Big|_{\Gamma_t}=0$$

- Since $\hat{w}_2^e=\hat{w}_1^{e+1}$

$$\hat{w}_1^1 f_1(\mathbf{K}^e,\mathbf{f}^e,\mathbf{u}^e)+\hat{w}_1^2 f_2(\mathbf{K}^e,\mathbf{f}^e,\mathbf{u}^e)+\ldots+\hat{w}_1^{N-1}f_{N-1}(\mathbf{K}^e,\mathbf{f}^e,\mathbf{u}^e)+\hat{w}_2^{N-1}f_N(\mathbf{K}^e,\mathbf{f}^e,\mathbf{u}^e)=0$$

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3.3 Finite element computation based on weak form

- Weak form in global domain

$$G(w,u)=\int_\Omega w(x)b_x\,\mathrm{d}x-\int_\Omega \frac{\partial w}{\partial x}E\frac{\partial u}{\partial x}\mathrm{d}x+w\bar{t}_x\Big|_{\Gamma_t}=0$$

- Weak form in each element

$$\hat{G}(\hat{w},\hat{u})=\hat{G}_\sigma(\hat{w},\hat{u})-\hat{G}_f(\hat{w},\hat{u})-\hat{w}(x)\bar{t}_x\Big|_{\Gamma_t}=0$$

- Since the parameters $\hat{w}_i^e$ are arbitrary, the expression multiplying each one must be zero. This leads to the set of equations:

$$\Rightarrow\begin{cases}f_1(\mathbf{K}^e,\mathbf{f}^e,\mathbf{u}^e)=0\\ f_2(\mathbf{K}^e,\mathbf{f}^e,\mathbf{u}^e)=0\\ \quad\vdots\\ f_{N-1}(\mathbf{K}^e,\mathbf{f}^e,\mathbf{u}^e)=0\\ f_N(\mathbf{K}^e,\mathbf{f}^e,\mathbf{u}^e)=0\end{cases}\Rightarrow\mathbf{Ku}-\mathbf{f}=\mathbf{0}\Rightarrow\mathbf{Ku}=\mathbf{f}$$

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3 Weak Form of Equivalent Integration

3.1 Weak form of equivalent integration for differential equations

3.2 Weak form of one-dimensional elasticity problems

3.3 Finite element computation based on weak form

3.4 Global assembly from one-dimensional elements

3.5 Treatments on boundary conditions

3.6 Exercises

3.4 Global assembly from one-dimensional elements

- In the one-dimensional beam model, N-1 elements ("M" small finite segments) and N nodes are used

- **Figure 3.5 Element and node numbers of one-dimensional problem.**

3.4 Global assembly from one-dimensional elements

- Then for each element we define the relationship of the local nodes to the global node number

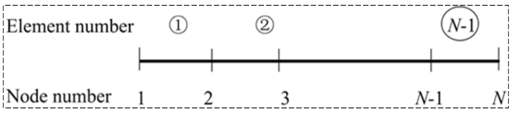
- **Table 3.2 Local to global node numbering for two-end element for one-dimensional elastic beam.**

Local node number	Element number	1	2	3	⋯	N-1
	1	1	2	3	⋯	N-1
	2	2	3	4	⋯	N

3.4 Global assembly from one-dimensional elements

- According to the entire node number of each element, the corresponding element location vector can be provided for determining the relationships between local and global node locations as follows

$$\lambda^1=\{1\quad 2\}^T \quad \lambda^2=\{2\quad 3\}^T \quad \cdots \quad \lambda^{N-1}=\{N-1\quad N\}^T$$



3.4 Global assembly from one-dimensional elements

- In order to implement global assembly from one-dimensional elements, the node numbers are marked on the left and upper sides of the matrix, and the expanded form of the stiffness matrix is given via element location vector

$$\begin{array}{c|cccccc} \text{Node number} & 1 & 2 & 3 & \cdots & N-1 & N \\ \hline 1 & K_{11}^1 & K_{12}^1 & 0 & & & \\ 2 & K_{21}^1 & K_{22}^1+K_{11}^2 & K_{12}^2 & 0 & & \\ 3 & 0 & K_{21}^2 & \left(K_{22}^2+K_{11}^3\right) & K_{12}^3 & & \\ \vdots & & & & \ddots & & \\ N-1 & & & & K_{21}^{N-1} & \left(K_{22}^{N-1}+K_{11}^N\right) & K_{12}^N \\ N & & & & & K_{21}^N & K_{22}^N \end{array} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \vdots \\ \hat{u}_{N-1} \\ \hat{u}_N \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-1} \\ f_N \end{Bmatrix}$$

3.4 Global assembly from one-dimensional elements

- Similarly, the of node numbers is marked on the left of the load vector, and the expanded load is given by

$$\begin{array}{c} \text{Node number} \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \end{array} \quad \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-1} \\ f_N \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1+f_1^2 \\ f_2^2+f_1^3 \\ \vdots \\ f_2^{N-2}+f_1^{N-1} \\ f_2^{N-1} \end{Bmatrix}$$

3.4 Global assembly from one-dimensional elements

- A standard linear problem with the final stiffness equations is as follows

$$\mathbf{Ku}=\mathbf{f}$$

$$\begin{array}{c|cccccc} \text{Node number} & 1 & 2 & 3 & \cdots & N-1 & N \\ \hline 1 & K_{11}^1 & K_{12}^1 & 0 & & & \\ 2 & K_{21}^1 & K_{22}^1+K_{11}^2 & K_{12}^2 & 0 & & \\ 3 & 0 & K_{21}^2 & \left(K_{22}^2+K_{11}^3\right) & K_{12}^3 & & \\ \vdots & & & & \ddots & & \\ N-1 & & & & K_{21}^{N-1} & \left(K_{22}^{N-1}+K_{11}^N\right) & K_{12}^N \\ N & & & & & K_{21}^N & K_{22}^N \end{array} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \vdots \\ \hat{u}_{N-1} \\ \hat{u}_N \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-1} \\ f_N \end{Bmatrix}$$

3.4 Global assembly from one-dimensional elements

- Matrix form of stiffness equation

$$\mathbf{Ku} = \mathbf{f}$$

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1N} \\ K_{21} & K_{22} & & K_{1N} \\ \vdots & & & \vdots \\ K_{N1} & K_{N2} & \cdots & K_{NN} \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{Bmatrix}$$

- Formal solution

$$\mathbf{u} = \mathbf{K}^{-1}\mathbf{f}$$

- Displacement and stress solution

$$\hat{u}^e(x') = N_1(x')\hat{u}_1^e + N_2(x')\hat{u}_2^e \qquad \hat{\sigma}_x^e(x') = E\frac{\partial \hat{u}^e(x')}{\partial x'} = E\left(\frac{\mathrm{d}N_1(x')}{\mathrm{d}x'}\hat{u}_1^e + \frac{\mathrm{d}N_2(x')}{\mathrm{d}x'}\hat{u}_2^e\right)$$

3 Weak Form of Equivalent Integration

3.5 Treatments on boundary conditions

- (1) Displacement boundary conditions $\hat{u}(x_a) \equiv \bar{u}_a$ $u(0) \equiv \bar{u}$ $\hat{u}_1 \equiv \bar{u}$

$$\begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 & & \\ 0 & K_{21}^2 & (K_{22}^2 + K_{11}^3) & K_{12}^3 & & \\ & & & \ddots & & \\ & & K_{21}^{N-1} & (K_{22}^{N-1} + K_{11}^N) & K_{12}^N & \\ & & & K_{21}^N & K_{22}^N & \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \vdots \\ \hat{u}_{N-1} \\ \hat{u}_N \end{Bmatrix} = \begin{Bmatrix} \bar{u}_1 \\ f_2 - K_{21}^1\bar{u}_1 \\ f_3 \\ \vdots \\ f_{N-1} \\ f_N \end{Bmatrix}$$

- (2) Traction boundary conditions $x = L$

$$f_N \to f_N + \bar{t}_x(L)$$

3.3 Finite element computation based on weak form

- Weak form in global domain

$$G(w, u) = \int_\Omega w(x)b_x \mathrm{d}x - \int_\Omega \frac{\partial w}{\partial x}E\frac{\partial u}{\partial x}\mathrm{d}x + w\bar{t}_x\big|_{\Gamma_t} = 0$$

- Weak form in each element

$$\hat{G}(\hat{w}, \hat{u}) = \hat{G}_\sigma(\hat{w}, \hat{u}) - \hat{G}_f(\hat{w}, \hat{u}) - \hat{w}(x)\bar{t}_x\big|_{\Gamma_t}$$

Review previous content

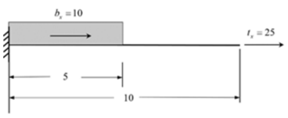
$$\hat{G}_\sigma(\hat{w}, \hat{u}) = \sum_{e=1}^{M}\left[\hat{w}_1^e \ \hat{w}_2^e\right]\int_0^{h_e}\begin{Bmatrix} \frac{\mathrm{d}N_1}{\mathrm{d}x'} \\ \frac{\mathrm{d}N_2}{\mathrm{d}x'} \end{Bmatrix} E_e \left[\frac{\mathrm{d}N_1}{\mathrm{d}x'} \quad \frac{\mathrm{d}N_2}{\mathrm{d}x'}\right]dx' \begin{Bmatrix} \hat{u}_1^e \\ \hat{u}_2^e \end{Bmatrix}$$

$$\hat{G}_f(\hat{w}, \hat{u}) = \sum_{e=1}^{M}\left[\hat{w}_1^e \quad \hat{w}_2^e\right]\int_0^{h_e}\begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} b_x \,\mathrm{d}x'$$

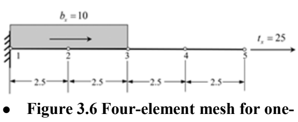
3.5 Treatments on boundary conditions

- **Example 3.2. Solutions of finite element method for one-dimensional elastic cantilever beam**

- As an example we consider a static problem with length 10 units and E=1000. Figure 3.1 shows the problem to solve.
- Loading $b_x = \begin{cases} 10 & \text{for } 0 < x < 5 \\ 0 & \text{for } 5 < x < 10 \end{cases}$
- Boundary conditions $u(0) = 0$ and $\bar{t}_x(10) = 25$
- We divide the domain into **four equal elements.**



- **Figure 3.1 One-dimensional elastic cantilever beam in Example 3.1.**



- **Figure 3.6 Four-element mesh for one-dimensional elastic cantilever beam in Example 3.2.**

3.3 Finite element computation based on weak form

- Each element can be evaluated as element stiffness matrix and load matrix

$$\mathbf{K}^e = \int_0^{h_e}\begin{Bmatrix} \frac{\mathrm{d}N_1}{\mathrm{d}x'} \\ \frac{\mathrm{d}N_2}{\mathrm{d}x'} \end{Bmatrix} E \left[\frac{\mathrm{d}N_1}{\mathrm{d}x'} \quad \frac{\mathrm{d}N_2}{\mathrm{d}x'}\right]dx' = \begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix}$$

$$\mathbf{f}^e = \int_0^{h_e}\begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} b_x \,\mathrm{d}x' = \begin{Bmatrix} f_1^e \\ f_2^e \end{Bmatrix}$$

Review previous content

- Element stiffness matrix and load matrix

E_e and b_x are constant $\quad N_1(x') = 1 - \frac{x'}{h_e}$ and $N_2(x') = \frac{x'}{h_e}$

$$\mathbf{K}^e = \frac{E_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{f}^e = \frac{1}{2}b_x h_e \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

3.5 Treatments on boundary conditions

- Element stiffness matrix and load matrix

$$\mathbf{K}^1=\mathbf{K}^2=\mathbf{K}^3=\mathbf{K}^4=\frac{E_e}{h_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=\frac{1000}{2.5}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=\begin{bmatrix}400 & -400\\ -400 & 400\end{bmatrix}$$

$$\mathbf{f}^1=\mathbf{f}^2=\frac{1}{2}b_x h_e\begin{Bmatrix}1\\1\end{Bmatrix}=\frac{1}{2}\times 10\times 2.5\begin{Bmatrix}1\\1\end{Bmatrix}=\begin{Bmatrix}12.5\\12.5\end{Bmatrix}$$

$$\mathbf{f}^3=\mathbf{f}^4=\frac{1}{2}b_x h_e\begin{Bmatrix}1\\1\end{Bmatrix}=\frac{1}{2}\times 0\times 2.5\begin{Bmatrix}1\\1\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

3.5 Treatments on boundary conditions

- Element stiffness matrix

Node number: 1, 2, 3, 4, 5

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & & & \\ K_{21}^1 & (K_{22}^1+K_{11}^2) & K_{12}^2 & & \\ & K_{21}^2 & (K_{22}^2+K_{11}^3) & K_{12}^3 & \\ & & K_{21}^3 & (K_{22}^3+K_{11}^4) & K_{12}^4 \\ & & & K_{21}^4 & K_{22}^4 \end{bmatrix}$$

$$=\begin{bmatrix} 400 & -400 & & & \\ -400 & 400+400 & -400 & & \\ & -400 & 400+400 & -400 & \\ & & -400 & 400+400 & -400 \\ & & & -400 & 400 \end{bmatrix}$$

$$=\begin{bmatrix} 400 & -400 & & & \\ -400 & 800 & -400 & & \\ & -400 & 800 & -400 & \\ & & -400 & 800 & -400 \\ & & & -400 & 400 \end{bmatrix}$$

3.5 Treatments on boundary conditions

- Load matrix

Node number: 1, 2, 3, 4, 5

$$\begin{Bmatrix} f_1\\ f_2\\ f_3\\ f_4\\ f_5 \end{Bmatrix}=\begin{Bmatrix} f_1^1\\ f_2^1+f_1^2\\ f_2^2+f_1^3\\ f_2^3+f_1^4\\ f_2^4 \end{Bmatrix}=\begin{Bmatrix} 12.5\\ 12.5+12.5\\ 12.5+0\\ 0+0\\ 0 \end{Bmatrix}=\begin{Bmatrix} 12.5\\ 25\\ 12.5\\ 0\\ 0 \end{Bmatrix}$$

- Matrix form of stiffness equation **Ku = f**

$$\begin{bmatrix} 400 & -400 & & & \\ -400 & 800 & -400 & & \\ & -400 & 800 & -400 & \\ & & -400 & 800 & -400 \\ & & & -400 & 400 \end{bmatrix}\begin{Bmatrix}\hat{u}_1\\ \hat{u}_2\\ \hat{u}_3\\ \hat{u}_4\\ \hat{u}_5\end{Bmatrix}=\begin{Bmatrix}12.5\\ 25\\ 12.5\\ 0\\ 0\end{Bmatrix}$$

3.5 Treatments on boundary conditions

- Displacement boundary conditions $\hat{u}_1\equiv\bar{u}_1\equiv 0$

$$\begin{bmatrix} 400 & -400 & & & \\ -400 & 800 & -400 & & \\ & -400 & 800 & -400 & \\ & & -400 & 800 & -400 \\ & & & -400 & 400 \end{bmatrix}\begin{Bmatrix}\hat{u}_1\\ \hat{u}_2\\ \hat{u}_3\\ \hat{u}_4\\ \hat{u}_5\end{Bmatrix}=\begin{Bmatrix}12.5\\ 25\\ 12.5\\ 0\\ 0\end{Bmatrix}$$

⇒

$$\begin{bmatrix} 1 & 0 & & & \\ 0 & 800 & -400 & & \\ & -400 & 800 & -400 & \\ & & -400 & 800 & -400 \\ & & & -400 & 400 \end{bmatrix}\begin{Bmatrix}\hat{u}_1\\ \hat{u}_2\\ \hat{u}_3\\ \hat{u}_4\\ \hat{u}_5\end{Bmatrix}=\begin{Bmatrix}0\\ 25\\ 12.5\\ 0\\ 0\end{Bmatrix}$$

$f_2-K_{21}^1\bar{u}_1$

3.5 Treatments on boundary conditions

- Traction boundary conditions $f_5\rightarrow f_5+\bar{t}_x(10)=25$

$$\begin{bmatrix} 800 & -400 & & \\ -400 & 800 & -400 & \\ & -400 & 800 & -400 \\ & & -400 & 400 \end{bmatrix}\begin{Bmatrix}\hat{u}_2\\ \hat{u}_3\\ \hat{u}_4\\ \hat{u}_5\end{Bmatrix}=\begin{Bmatrix}25\\ 12.5\\ 0\\ 0\end{Bmatrix}$$

⇒

$$\begin{bmatrix} 800 & -400 & & \\ -400 & 800 & -400 & \\ & -400 & 800 & -400 \\ & & -400 & 400 \end{bmatrix}\begin{Bmatrix}\hat{u}_2\\ \hat{u}_3\\ \hat{u}_4\\ \hat{u}_5\end{Bmatrix}=\begin{Bmatrix}25\\ 12.5\\ 0\\ 25\end{Bmatrix}$$

3.5 Treatments on boundary conditions

- Formal solution $\mathbf{u}=\mathbf{K}^{-1}\mathbf{f}$

$$\begin{bmatrix} 800 & -400 & & \\ -400 & 800 & -400 & \\ & -400 & 800 & -400 \\ & & -400 & 400 \end{bmatrix}\begin{Bmatrix}\hat{u}_2\\ \hat{u}_3\\ \hat{u}_4\\ \hat{u}_5\end{Bmatrix}=\begin{Bmatrix}0\\ 12.5\\ 0\\ 25\end{Bmatrix}\Rightarrow\begin{Bmatrix}\hat{u}_2\\ \hat{u}_3\\ \hat{u}_4\\ \hat{u}_5\end{Bmatrix}=\begin{Bmatrix}0.15625\\ 0.25\\ 0.3125\\ 0.375\end{Bmatrix}$$

- Displacement and stress solution

$$\hat{u}^e(x')=N_1(x')\hat{u}_1^e+N_2(x')\hat{u}_2^e \qquad \hat{\sigma}_x^e(x')=E\frac{\partial\hat{u}^e(x')}{\partial x'}=E\left(\frac{\mathrm{d}N_1(x')}{\mathrm{d}x'}\hat{u}_1^e+\frac{\mathrm{d}N_2(x')}{\mathrm{d}x'}\hat{u}_2^e\right)$$

$$\hat{u}^1(x')=\left(1-\frac{x'}{h_e}\right)\times 0+\frac{x'}{h_e}\times 0.15625=0.0625x'$$

$$\hat{\sigma}_x^1(x')=E\frac{\partial\hat{u}^1(x')}{\partial x'}=E\left(-\frac{1}{h_e}\times 0+\frac{1}{h_e}\times 0.15625\right)=62.5$$

3.5 Treatments on boundary conditions

- Displacement and stress solution

$$\hat{u}^e(x') = N_1(x')\hat{u}_1^e + N_2(x')\hat{u}_2^e \qquad \hat{\sigma}_x^e(x') = E\frac{\partial \hat{u}^e(x')}{\partial x'} = E\left(\frac{\mathrm{d}N_1(x')}{\mathrm{d}x'}\hat{u}_1^e + \frac{\mathrm{d}N_2(x')}{\mathrm{d}x'}\hat{u}_2^e\right)$$

$$\hat{u}^2(x') = \left(1 - \frac{x'}{h_e}\right)\times 0.15625 + \frac{x'}{h_e}\times 0.25 = 0.0375x' + 0.15625$$

$$\hat{\sigma}_x^2(x') = E\frac{\partial \hat{u}^2(x')}{\partial x'} = E\left(-\frac{1}{h_e}\times 0.15625 + \frac{1}{h_e}\times 0.25\right) = 37.5$$

$$\hat{u}^3(x') = \left(1 - \frac{x'}{h_e}\right)\times 0.25 + \frac{x'}{h_e}\times 0.3125 = 0.025x' + 0.25$$

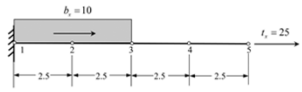
$$\hat{\sigma}_x^3(x') = E\frac{\partial \hat{u}^3(x')}{\partial x'} = E\left(-\frac{1}{h_e}\times 0.25 + \frac{1}{h_e}\times 0.3125\right) = 25$$

$$\hat{u}^4(x') = \left(1 - \frac{x'}{h_e}\right)\times 0.3125 + \frac{x'}{h_e}\times 0.375 = 0.025x' + 0.3125$$

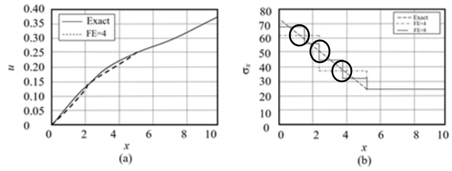
$$\hat{\sigma}_x^4(x') = E\frac{\partial \hat{u}^4(x')}{\partial x'} = E\left(-\frac{1}{h_e}\times 0.3125 + \frac{1}{h_e}\times 0.375\right) = 25$$

3.5 Treatments on boundary conditions

- We divide the domain into four equal elements.
- Stresses are converging more slowly than the displacements (this was true for the first example also) but again have *super-convergent* points.



- **Figure 3.6 Four-element mesh for one-dimensional elastic cantilever beam in Example 3.2.**



- **Figure 3.7 Displacement and stress solutions in Example 3.2 based on finite element method using *N*-element solutions: (a) displacement and (b) stress.**

Galerkin method VS Finite element method

- **Similarities**

  1 Weak form.

  2 Solution function.

  3 Algebraic equations.

- **Differences**

  1 Domain of solution function: Element, shape function.

  2 Treatments on boundary conditions: **Ku=f**.

The End