4 Elements and Shape Functions

■ **Contents**

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■ **Keywords**

- **Lagrange element** 拉格朗日单元
- **Triangle** 三角形 **Triangle element** 三角形单元
- **Rectangle** 四边形 **Rectangle element** 四边形单元
- **Tetrahedron** 四面体 **Tetrahedron element** 四面体单元
- **Hexagon** 六面体 **Hexagon element** 六面体单元
- **Linear element** 线性单元
- **Quadratic element** 二次单元

4 Elements and Shape Functions

4.1 One-dimensional Lagrange element

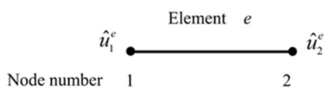
■ **One-dimensional elastic cantilever beam**

- To compute the element matrices, it is necessary to devise appropriate shape functions.
- Previously, we introduced one-dimensional element with two nodes, which is actually the simplest low order linear element. We will also introduce high order Lagrange element for one-dimensional problem in this chapter.
- Further, for two-dimensional problems, we consider the simplest form for elements of triangular and rectangular form; for three-dimensional problems, the simple tetrahedron and hexahedron element form are also developed.

4.1 One-dimensional Lagrange element

■ **4.1.1 Linear element with two nodes**

- This element of line with two nodes has only two end nodes as shown in Fig. 4.1, and the corresponding two displacements on the nodes are $\hat{u}_1^e$ and $\hat{u}_2^e$. This kind of element is called the one-dimensional element with two nodes.



- **Figure 4.1 One-dimensional element with two nodes.**

4.1 One-dimensional Lagrange element

- For this one-dimensional problem, we write the displacement as

$$u^e \approx \hat{u}^e = \sum_{a=1}^{2} N_a(x')\hat{u}_a^e = N_1(x')\hat{u}_1^e + N_2(x')\hat{u}_2^e$$

In the previous introduction, we define a local coordinate system in each element as $x' = x - x_1^e$ and the element length as $h_e = x_2^e - x_1^e$, the shape functions are given by

$$N_1(x') = 1 - \frac{x'}{h_e} \quad \text{and} \quad N_2(x') = \frac{x'}{h_e}$$

- The displacement function of the element is linear of x, therefore it is called one-dimensional linear element.

4.1 One-dimensional Lagrange element

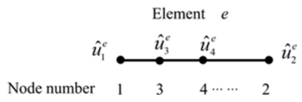
- The derivatives of the shape functions are given by

$$\frac{\mathrm{d}N_1}{\mathrm{d}x}=\frac{\mathrm{d}N_1}{\mathrm{d}x'}=-\frac{1}{h_e}\quad \text{and}\quad \frac{\mathrm{d}N_2}{\mathrm{d}x}=\frac{\mathrm{d}N_2}{\mathrm{d}x'}=\frac{1}{h_e}$$

$$N_1(x')=1-\frac{x'}{h_e}\quad \text{and}\quad N_2(x')=\frac{x'}{h_e}$$

4.1 One-dimensional Lagrange element

- **4.1.2 Higher order Lagrange element**
- The above linear element is difficult to describe the complex changes of displacement in the element. Later, higher order shape functions and more nodes were developed to represent the displacements of the element. The higher order element with more nodes is shown in Fig. 4.2.



- **Figure 4.2 One-dimensional higher order element.**

4.1 One-dimensional Lagrange element

- The displacement is written as

$$u^e\approx\hat{u}^e=\sum_{a=1}^{n}N_a(\xi)\,\hat{u}_a^e=N_1(\xi)\,\hat{u}_1^e+N_2(\xi)\,\hat{u}_2^e+\ldots+N_n(\xi)\,\hat{u}_n^e\qquad -1\le\xi\le 1$$

where n is the total number of functions used in the element; $\hat{u}_a^e$ are the unknown parameters to be determined; ξ is a new local coordinate, the reason for this choice will become clear later when we describe the use of numerical integration to evaluate the integrals in weak form.

4.1 One-dimensional Lagrange element

- The local coordinate ξ possesses the relationship with global coordinate x

$$x=\sum_{a=1}^{n}N_a(\xi)\,x_a^e=N_1(\xi)\,x_1^e+N_2(\xi)\,x_2^e+\ldots+N_n(\xi)\,x_n^e\qquad -1\le\xi\le 1$$

The global coordinate and local coordinates in the above coordinate transformation relation shows one-to-one correspondence, but it is not necessarily linear. Only when the shape functions are taken as linear functions (such as the shape functions of two node elements obtained previously) has linear relationship.

4.1 One-dimensional Lagrange element

- In order to ensure that the unknown parameters to be solved are the displacements on nodes, the requirement is that

$$N_a(\xi_b)=\delta_{ab}=\begin{cases}1, & \xi_a=\xi_b\\ 0, & \xi_a\ne\xi_b\end{cases}$$

where ξ_b is the local coordinate which has position x_b.

4.1 One-dimensional Lagrange element

- To satisfy the above equation, a simple construction for higher order shape functions is the Lagrange interpolation formula given by

$$l_a^p(\xi)=\prod_{\substack{b=1\\ b\ne a}}^{n}\frac{(\xi-\xi_b)}{(\xi_a-\xi_b)}$$
$$=\frac{(\xi-\xi_1)(\xi-\xi_2)(\cdots)(\xi-\xi_{a-1})(\xi-\xi_{a+1})(\cdots)(\xi-\xi_n)}{(\xi_a-\xi_1)(\xi_a-\xi_2)(\cdots)(\xi_a-\xi_{a-1})(\xi_a-\xi_{a+1})(\cdots)(\xi_a-\xi_n)}$$

where the order of the polynomial is $p=n-1$. Once we choose the location for the end nodes, the internal values of ξ_a may be spaced at uniform increments.

4.1 One-dimensional Lagrange element

- For one-dimensional elements we can set

$$\boxed{N_a(\xi) = l_a^p(\xi)}$$

to define the shape functions. The completeness condition then requires that $\hat{u}^e(\xi)$ contain any constant c (displacement of rigid body), which then yields

$$\hat{u}^e(\xi) = \sum_{a=1}^{n} N_a(\xi)\, c = c \quad \text{or} \quad \sum_{a=1}^{n} N_a(\xi) = 1$$

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4.1 One-dimensional Lagrange element

- Especially, in the linear example above, we set $\xi_1 = -1$ and $\xi_2 = 1$ with n=2 (p=1), then we can obtain the two shape functions

$$N_1(\xi) = l_1^1(\xi) = \frac{\xi - 1}{-1-1} = \frac{1}{2}(1-\xi)$$

$$N_2(\xi) = l_2^1(\xi) = \frac{\xi + 1}{1+1} = \frac{1}{2}(1+\xi)$$

- The global coordinate x can be expressed by local coordinate as ξ

$$x = \sum_{a=1}^{2} N_a(\xi)\, x_a^e = N_1(\xi)\, x_1^e + N_2(\xi)\, x_2^e = \frac{1}{2}(1-\xi)\, x_1^e + \frac{1}{2}(1+\xi)\, x_2^e$$

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4.1 One-dimensional Lagrange element

- The derivative of the coordinate function is given by

$$j_e = \frac{\partial x}{\partial \xi} = \frac{h_e}{2}$$

The above derivative between the global and the local coordinates is **Jacobian**, which is represented as j_e.

- The derivatives of the shape functions are given by

$$\frac{\partial N_a}{\partial x} = \frac{1}{j_e(\xi)} \frac{\partial N_a}{\partial \xi}, \quad a = 1, 2$$

$$\frac{\partial N_1}{\partial x} = \frac{1}{j_e} \frac{\partial N_1}{\partial \xi} = -\frac{1}{h_e} \qquad \frac{\partial N_2}{\partial x} = \frac{1}{j_e} \frac{\partial N_2}{\partial \xi} = \frac{1}{h_e}$$

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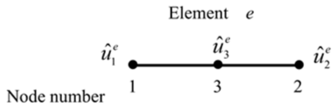
4.1 One-dimensional Lagrange element

■ 4.1.3 Quadratic Lagrange element

- For quadratic shape functions we let the nodes be placed at

$$\xi_1 = -1, \quad \xi_2 = 1, \quad \text{and} \quad \xi_3 = 0$$

in which the order of the polynomial is p=3−1=2 (n=3). The quadratic Lagrange element with three nodes is shown in Fig. 4.3



- **Figure 4.3 One-dimensional quadratic Lagrange element with three nodes.**

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4.1 One-dimensional Lagrange element

- We obtain the three shape functions using Lagrange interpolation formula as

$$N_1(\xi) = l_1^2(\xi) = \frac{(\xi-1)(\xi-0)}{(-1-1)(-1-0)} = \frac{1}{2}\xi(\xi-1)$$

$$N_2(\xi) = l_2^2(\xi) = \frac{(\xi+1)(\xi-0)}{(1+1)(1-0)} = \frac{1}{2}\xi(\xi+1)$$

$$N_3(\xi) = l_3^2(\xi) = \frac{(\xi+1)(\xi-1)}{(0+1)(0-1)} = 1 - \xi^2$$

$$l_a^p(\xi) = \prod_{\substack{b=1 \\ b \neq a}}^{n} \frac{(\xi - \xi_b)}{(\xi_a - \xi_b)} = \frac{(\xi-\xi_1)(\xi-\xi_2)(\cdots)(\xi-\xi_{a-1})(\xi-\xi_{a+1})(\cdots)(\xi-\xi_n)}{(\xi_a-\xi_1)(\xi_a-\xi_2)(\cdots)(\xi_a-\xi_{a-1})(\xi_a-\xi_{a+1})(\cdots)(\xi_a-\xi_n)}$$

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4.1 One-dimensional Lagrange element

- If we let the global coordinates for the element be given by x_1^e, x_2^e, and x_3^e (x_1^e and x_2^e are the boundary end nodes), the global coordinate x can be expressed by local coordinate ξ as

$$x = N_1(\xi)x_1^e + N_2(\xi)x_2^e + N_3(\xi)x_3^e$$

- The Jacobian is now given by

$$j_e(\xi) = \frac{\partial x}{\partial \xi} = \left(\xi - \frac{1}{2}\right)x_1^e + \left(\xi + \frac{1}{2}\right)x_2^e - 2\xi x_3^e = \frac{1}{2}h_e + \xi\left(x_1^e + x_2^e - 2x_3^e\right)$$

the Jacobian is not constant unless the coordinate x_3^e for node 3 is placed at the middle of the element.

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4.1 One-dimensional Lagrange element

- The derivatives of the shape functions are given by

$$\frac{\partial N_a}{\partial x} = \frac{1}{j_e(\xi)}\frac{\partial N_a}{\partial \xi}, \quad a = 1, 2, ..., n$$

$$N_1(\xi) = l_1^2(\xi) = \frac{(\xi-1)(\xi-0)}{(-1-1)(-1-0)} = \frac{1}{2}\xi(\xi-1)$$

$$N_2(\xi) = l_2^2(\xi) = \frac{(\xi+1)(\xi-0)}{(1+1)(1-0)} = \frac{1}{2}\xi(\xi+1)$$

$$N_3(\xi) = l_3^2(\xi) = \frac{(\xi+1)(\xi-1)}{(0+1)(0-1)} = 1-\xi^2$$

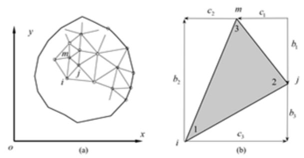
$$j_e(\xi) = \frac{\partial x}{\partial \xi} = \left(\xi - \frac{1}{2}\right)x_1^e + \left(\xi + \frac{1}{2}\right)x_2^e - 2\xi x_3^e = \frac{1}{2}h_e + \xi\left(x_1^e + x_2^e - 2x_3^e\right)$$

4 Elements and Shape Functions

4.2 Two-dimensional triangle element

■ **Triangle with three nodes**

- The finite element domain is defined by dividing the domain into a mesh of two-dimensional triangular elements as shown in Fig. 4.4a. A simple set of linear functions can be constructed from linear polynomials over three-node triangles as shown in Fig. 4.4b.



- **Figure 4.4 Division of a two-dimensional domain into triangle elements: (a) triangle mesh, (b) triangle element.**

4.2 Two-dimensional triangle element

- The approximation in each triangle may be written as a linear function of the Cartesian coordinates:

$$\phi^e \approx \hat{\phi}^e = \begin{bmatrix} 1 & x & y \end{bmatrix}\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \alpha_1 + \alpha_2 x + \alpha_3 y$$

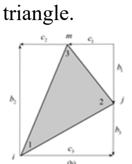
where ϕ^e is displacement on the element, which can be the displacement u in x-direction or the displacement v in y-direction; the unknown parameters α_1 to α_3 may be evaluated in terms of the displacements at each of the three vertices of the triangle.

4.2 Two-dimensional triangle element

- The vertices define the nodes of the triangle. Accordingly, we write the set of equations

$$\begin{Bmatrix} \hat{\phi}_1^e \\ \hat{\phi}_2^e \\ \hat{\phi}_3^e \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

where x_a and y_a are coordinates at the three vertices of the triangle.



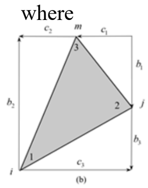
$$\phi^e \approx \hat{\phi}^e = \begin{bmatrix} 1 & x & y \end{bmatrix}\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \alpha_1 + \alpha_2 x + \alpha_3 y$$

4.2 Two-dimensional triangle element

- The inverse to the coefficient matrix is given by

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} = \frac{1}{2\Delta}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

where



$$a_1 = x_2 y_3 - x_3 y_2, \quad b_1 = y_2 - y_3, \quad c_1 = x_3 - x_2$$
$$a_2 = x_3 y_1 - x_1 y_3, \quad b_2 = y_3 - y_1, \quad c_2 = x_1 - x_3$$
$$a_3 = x_1 y_2 - x_2 y_1, \quad b_3 = y_1 - y_2, \quad c_3 = x_2 - x_1$$

area of the triangle $\Delta = (x_1 b_1 + x_2 b_2 + x_3 b_3)/2$

4.2 Two-dimensional triangle element

- By substitution, we can get

$$\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix}=\begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{bmatrix}^{-1}\begin{Bmatrix}\hat{\phi}_1^e\\ \hat{\phi}_2^e\\ \hat{\phi}_3^e\end{Bmatrix}=\frac{1}{2\Delta}\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\begin{Bmatrix}\hat{\phi}_1^e\\ \hat{\phi}_2^e\\ \hat{\phi}_3^e\end{Bmatrix}=\frac{1}{2\Delta}\begin{Bmatrix}\sum_{a=1}^{3} a_a\hat{\phi}_a^e\\ \sum_{a=1}^{3} b_a\hat{\phi}_a^e\\ \sum_{a=1}^{3} c_a\hat{\phi}_a^e\end{Bmatrix}$$

- The above solution for the parameters α_a permits the element interpolations to be rewritten in terms of nodal parameters as

$$\begin{aligned}\hat{\phi}^e &= \alpha_1+\alpha_2 x+\alpha_3 y\\ &=\sum_{a=1}^{3}\frac{1}{2\Delta}\left(a_a\hat{\phi}_a+b_a\hat{\phi}_a x+c_a\hat{\phi}_a y\right)\\ &=\sum_{a=1}^{3}\frac{1}{2\Delta}\left(a_a+b_a x+c_a y\right)\hat{\phi}_a^e\end{aligned}$$

$$\phi^e\approx\hat{\phi}^e=\begin{bmatrix}1 & x & y\end{bmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix}=\alpha_1+\alpha_2 x+\alpha_3 y$$

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4.2 Two-dimensional triangle element

- Thus, the three shape functions for the triangle are given by

$$N_a(x,y)=\frac{1}{2\Delta}\left(a_a+b_a x+c_a y\right),\quad a=1,2,3$$

$$\begin{aligned}\hat{\phi}^e &= \alpha_1+\alpha_2 x+\alpha_3 y\\ &=\sum_{a=1}^{3}\frac{1}{2\Delta}\left(a_a\hat{\phi}_a+b_a\hat{\phi}_a x+c_a\hat{\phi}_a y\right)\\ &=\sum_{a=1}^{3}\frac{1}{2\Delta}\left(a_a+b_a x+c_a y\right)\hat{\phi}_a^e\end{aligned}$$

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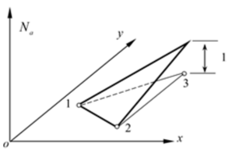
4.2 Two-dimensional triangle element

- The shape function for a=3 is shown in Fig. 4.5a., it can be seen that the value of this shape function is one at node 3, and the value is zero at other nodes 1 or 2. More generally, these shape functions have the following properties



$$N_a(x_b,y_b)=\delta_{ab}=\begin{cases}1, & a=b\\ 0, & a\neq b\end{cases}$$

where (x_b, y_b) is local coordinate.

- **Figure 4.5 Shape function N_3 for two-dimensional triangle with three nodes.**

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4.2 Two-dimensional triangle element

- Besides, the completeness condition then requires that $\hat{\phi}^e(x_b, y_b)$ contains any constant c (displacement of rigid body), which then yields

$$\hat{\phi}^e(x,y)=\sum_{a=1}^{3}N_a(x,y)\,c=c\quad\text{or}\quad\sum_{a=1}^{3}N_a(x,y)=1$$

- Using these shape functions, we can write the set of approximations in each individual element as

$$\hat{\phi}^e=\sum_{a=1}^{3}N_a(x,y)\hat{\phi}_a^e$$

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The End