Computational Mechanics

(*English Course*)

Yongliang Wang

School of Mechanics and Civil Engineering

China University of Mining and Technology (Beijing)

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4 Elements and Shape Functions

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4 Elements and Shape Functions

■ **Keywords**

- **Lagrange element** 拉格朗日单元
- **Triangle** 三角形 **Triangle element** 三角形单元
- **Rectangle** 四边形 **Rectangle element** 四边形单元
- **Tetrahedron** 四面体 **Tetrahedron element** 四面体单元
- **Hexagon** 六面体 **Hexagon element** 六面体单元
- **Area coordinate** 面积坐标 **Volume coordinate** 体积坐标
- **Linear element** 线性单元
- **Quadratic element** 二次单元
- **Cubic element** 三次单元

4 Elements and Shape Functions

4.1 One-dimensional Lagrange element

4.2 Two-dimensional triangle element

4.3 Two-dimensional rectangle element

4.4 Three-dimensional tetrahedron element

4.5 Three-dimensional hexahedron element

4.6 Exercises

4.1 One-dimensional Lagrange element

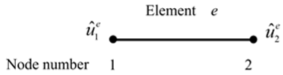
■ **One-dimensional elastic cantilever beam**

- To compute the element matrices, it is necessary to select appropriate shape functions.
- Previously, we introduced one-dimensional element with two nodes, which is actually the simplest low order linear element. We will also introduce high-order Lagrange element for one-dimensional problem in this chapter.
- Further, for two-dimensional problems, we consider the triangle and rectangle elements are provided; for three-dimensional problems, the tetrahedron and hexahedron element are also presented.

4.1 One-dimensional Lagrange element

■ **4.1.1 Linear element with two nodes**

- This element of line with two nodes has only two end nodes as shown in Fig. 4.1, and the corresponding two displacements on the nodes are $\hat{u}_1^e$ and $\hat{u}_2^e$. This kind of element is called the one-dimensional element with two nodes.



- **Figure 4.1 One-dimensional element with two nodes.**

4.1 One-dimensional Lagrange element

- For this one-dimensional problem, we write the displacement as

$$u^e \approx \hat{u}^e = \sum_{a=1}^{2} N_a(x')\,\hat{u}_a^e = N_1(x')\,\hat{u}_1^e + N_2(x')\,\hat{u}_2^e$$

In the previous introduction, we define a local coordinate system in each element as $x' = x - x_1^e$ and the element length as $h_e = x_2^e - x_1^e$, the shape functions are given by

$$N_1(x') = 1 - \frac{x'}{h_e} \quad \text{and} \quad N_2(x') = \frac{x'}{h_e}$$

- The displacement function of the element is linear of x, therefore it is called one-dimensional linear element.

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4.1 One-dimensional Lagrange element

- The derivatives of the shape functions are given by

$$\frac{\mathrm{d}N_1}{\mathrm{d}x} = \frac{\mathrm{d}N_1}{\mathrm{d}x'} = -\frac{1}{h_e} \quad \text{and} \quad \frac{\mathrm{d}N_2}{\mathrm{d}x} = \frac{\mathrm{d}N_2}{\mathrm{d}x'} = \frac{1}{h_e}$$

$$N_1(x') = 1 - \frac{x'}{h_e} \quad \text{and} \quad N_2(x') = \frac{x'}{h_e}$$

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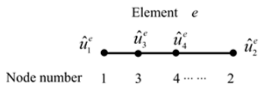
4.1 One-dimensional Lagrange element

4.1.2 Higher order Lagrange element

- The above linear element is difficult to describe the complex changes of displacement in the element. Later, higher order shape functions and more nodes were developed to represent the displacements of the element. The higher order element with more nodes is shown in Fig. 4.2.



- **Figure 4.2 One-dimensional higher order element.**

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4.1 One-dimensional Lagrange element

- The displacement is written as

$$u^e \approx \hat{u}^e = \sum_{a=1}^{n} N_a(\xi)\,\hat{u}_a^e = N_1(\xi)\,\hat{u}_1^e + N_2(\xi)\,\hat{u}_2^e + \ldots + N_n(\xi)\,\hat{u}_n^e \quad -1 \le \xi \le 1$$

where n is the total number of functions used in the element; $\hat{u}_a^e$ are the unknown parameters to be determined; ξ is a new local coordinate, the reason for this choice will become clear later when we describe the use of numerical integration to evaluate the integrals in weak form.

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4.1 One-dimensional Lagrange element

- The local coordinate ξ possesses the relationship with global coordinate x

$$x = \sum_{a=1}^{n} N_a(\xi)\,x_a^e = N_1(\xi)\,x_1^e + N_2(\xi)\,x_2^e + \ldots + N_n(\xi)\,x_n^e \quad -1 \le \xi \le 1$$

The global coordinate and local coordinates in the above coordinate transformation relation shows one-to-one correspondence, but it is not necessarily linear. Only when the shape functions are taken as linear functions (such as the shape functions of two node elements obtained previously) has linear relationship.

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4.1 One-dimensional Lagrange element

- In order to ensure that the unknown parameters to be solved are the displacements on nodes, the requirement is that

$$N_a(\xi_b) = \delta_{ab} = \begin{cases} 1, & \xi_a = \xi_b \\ 0, & \xi_a \ne \xi_b \end{cases}$$

where ξ_b is the local coordinate which has position x_b.

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4.1 One-dimensional Lagrange element

- To satisfy the above equation, a simple construction for higher order shape functions is the Lagrange interpolation formula given by

$$l_a^p(\xi)=\prod_{\substack{b=1\\b\neq a}}^{n}\frac{(\xi-\xi_b)}{(\xi_a-\xi_b)}$$
$$=\frac{(\xi-\xi_1)(\xi-\xi_2)(\cdots)(\xi-\xi_{a-1})(\xi-\xi_{a+1})(\cdots)(\xi-\xi_n)}{(\xi_a-\xi_1)(\xi_a-\xi_2)(\cdots)(\xi_a-\xi_{a-1})(\xi_a-\xi_{a+1})(\cdots)(\xi_a-\xi_n)}$$

where the order of the polynomial is $p=n-1$. Once we choose the location for the end nodes, the internal values of ξ_a may be spaced at uniform increments.

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4.1 One-dimensional Lagrange element

- For one-dimensional elements we can set

$$\boxed{N_a(\xi)=l_a^p(\xi)}$$

to define the shape functions. The completeness condition then requires that $\hat{u}^e(\xi)$ contains any constant *c* (displacement of rigid body), which then yields

$$\hat{u}^e(\xi)=\sum_{a=1}^{n}N_a(\xi)c=c \quad \text{or} \quad \sum_{a=1}^{n}N_a(\xi)=1$$

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4.1 One-dimensional Lagrange element

- Especially, in the linear example above, we set $\xi_1=-1$ and $\xi_2=1$ with *n*=2 (*p*=1), then we can obtain the two shape functions

$$N_1(\xi)=l_1^1(\xi)=\frac{\xi-1}{-1-1}=\frac{1}{2}(1-\xi)$$
$$N_2(\xi)=l_2^1(\xi)=\frac{\xi+1}{1+1}=\frac{1}{2}(1+\xi)$$

- The global coordinate *x* can be expressed by local coordinate as ξ

$$x=\sum_{a=1}^{2}N_a(\xi)x_a^e=N_1(\xi)x_1^e+N_2(\xi)x_2^e=\frac{1}{2}(1-\xi)x_1^e+\frac{1}{2}(1+\xi)x_2^e$$

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4.1 One-dimensional Lagrange element

- The derivative of the coordinate function is given by

$$j_e=\frac{\partial x}{\partial \xi}=\frac{h_e}{2}$$

The above derivative between the global and the local coordinates is **Jacobian**, which is represented as j_e .

- The derivatives of the shape functions are given by

$$\frac{\partial N_a}{\partial x}=\frac{1}{j_e(\xi)}\frac{\partial N_a}{\partial \xi},\quad a=1,2$$

$$\frac{\partial N_1}{\partial x}=\frac{1}{j_e}\frac{\partial N_1}{\partial \xi}=-\frac{1}{h_e} \qquad \frac{\partial N_2}{\partial x}=\frac{1}{j_e}\frac{\partial N_2}{\partial \xi}=\frac{1}{h_e}$$

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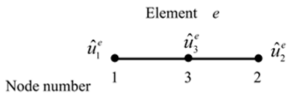
4.1 One-dimensional Lagrange element

- **4.1.3 Quadratic Lagrange element**
- For quadratic shape functions we let the nodes be placed at

$$\xi_1=-1,\quad \xi_2=1,\quad \text{and}\quad \xi_3=0$$

in which the order of the polynomial is $p=3-1=2$ ($n=3$). The quadratic Lagrange element with three nodes is shown in Fig. 4.3



- **Figure 4.3 One-dimensional quadratic Lagrange element with three nodes.**

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4.1 One-dimensional Lagrange element

- We obtain the three shape functions using Lagrange interpolation formula as

$$N_1(\xi)=l_1^2(\xi)=\frac{(\xi-1)(\xi-0)}{(-1-1)(-1-0)}=\frac{1}{2}\xi(\xi-1)$$
$$N_2(\xi)=l_2^2(\xi)=\frac{(\xi+1)(\xi-0)}{(1+1)(1-0)}=\frac{1}{2}\xi(\xi+1)$$
$$N_3(\xi)=l_3^2(\xi)=\frac{(\xi+1)(\xi-1)}{(0+1)(0-1)}=1-\xi^2$$

$$l_a^p(\xi)=\prod_{\substack{b=1\\b\neq a}}^{n}\frac{(\xi-\xi_b)}{(\xi_a-\xi_b)}$$
$$=\frac{(\xi-\xi_1)(\xi-\xi_2)(\cdots)(\xi-\xi_{a-1})(\xi-\xi_{a+1})(\cdots)(\xi-\xi_n)}{(\xi_a-\xi_1)(\xi_a-\xi_2)(\cdots)(\xi_a-\xi_{a-1})(\xi_a-\xi_{a+1})(\cdots)(\xi_a-\xi_n)}$$

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4.1 One-dimensional Lagrange element

- If we let the global coordinates for the element be given by x_1^e, x_2^e, and x_3^e (x_1^e and x_2^e are the boundary end nodes), the global coordinate x can be expressed by local coordinate ξ as

$$x = N_1(\xi)x_1^e + N_2(\xi)x_2^e + N_3(\xi)x_3^e$$

- The Jacobian is now given by

$$j_e(\xi) = \frac{\partial x}{\partial \xi} = \left(\xi - \frac{1}{2}\right)x_1^e + \left(\xi + \frac{1}{2}\right)x_2^e - 2\xi x_3^e = \frac{1}{2}h_e + \xi\left(x_1^e + x_2^e - 2x_3^e\right)$$

the Jacobian is not constant unless the coordinate x_3^e for node 3 is placed at the middle of the element.

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4.1 One-dimensional Lagrange element

- The derivatives of the shape functions are given by

$$\frac{\partial N_a}{\partial x} = \frac{1}{j_e(\xi)}\frac{\partial N_a}{\partial \xi}, \quad a = 1, 2, ..., n$$

$$N_1(\xi) = l_1^2(\xi) = \frac{(\xi-1)(\xi-0)}{(-1-1)(-1-0)} = \frac{1}{2}\xi(\xi-1)$$

$$N_2(\xi) = l_2^2(\xi) = \frac{(\xi+1)(\xi-0)}{(1+1)(1-0)} = \frac{1}{2}\xi(\xi+1)$$

$$N_3(\xi) = l_3^2(\xi) = \frac{(\xi+1)(\xi-1)}{(0+1)(0-1)} = 1-\xi^2$$

$$j_e(\xi) = \frac{\partial x}{\partial \xi} = \left(\xi - \frac{1}{2}\right)x_1^e + \left(\xi + \frac{1}{2}\right)x_2^e - 2\xi x_3^e = \frac{1}{2}h_e + \xi\left(x_1^e + x_2^e - 2x_3^e\right)$$

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The End

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