## **4** Elements and Shape Functions

4.1 One-dimensional Lagrange element

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- 4.2 Two-dimensional triangle element
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#### 4.3 Two-dimensional rectangle element

- Linear rectangle element with four nodes
- As a second example of two-dimensional shape functions, we consider rectangles of the form shown in Fig. 4.10. The rectangular element considered has side lengths of 2*a* and 2*b* in the *x* and *y*-directions, respectively.



## 4.3 Two-dimensional rectangle element

• For the derivation of the shape functions it is convenient to use a local Cartesian system *x'*, *y'* defined by

$$x' = x - x_0$$
 and  $y' = y - y_0$ 

 $x_0 = \frac{1}{4} \sum_{\alpha=1}^{4} x_{\alpha} \text{ and } y_0 = \frac{1}{4} \sum_{\alpha=1}^{4} y_{\alpha}$ in which  $x_0, y_0$  are located at the center of the rectangle and  $x_a, y_a$  are

coordinates of the nodes.

• Figure 4.10 Rectangle element geometry and node numbers.

## 4.3 Two-dimensional rectangle element

 $\phi^{\epsilon}$ 

• We now need four functions for each displacement component in order to uniquely define the shape functions. A suitable choice is given by:

$$\approx \hat{\phi}^{*} = \begin{bmatrix} 1 & x' & y' & x'y' \end{bmatrix} \begin{cases} \alpha_{1} \\ \alpha_{2} \\ \alpha_{3} \\ \alpha_{4} \end{cases} = \alpha_{1} + x'\alpha_{2} + y'\alpha_{3} + x'y'\alpha_{4}$$

where  $\phi^{e}$  is displacement on the element, which can be the displacement *u* in *x*-direction or the displacement *v* in *y*-direction; the unknown parameters  $\alpha_1$  to  $\alpha_4$  may be evaluated in terms of the displacements at each of the four vertices of the rectangle.











# 4.3 Two-dimensional rectangle element

• More generally, these shape functions have the following properties:

$$N_a(\xi_b, \eta_b) = \delta_{ab} = \begin{cases} 1, & a = b \\ 0, & a \neq b \end{cases}$$

where  $(\xi_b, \eta_b)$  is the local coordinate.

Besides, the completeness condition then requires that φ<sup>\*</sup>(ξ, η) contains any constant c (displacement of rigid body), which then yields

$$\hat{\phi}^{*}(\xi,\eta) = \sum_{a=1}^{4} N_{a}(\xi,\eta) c = c \quad \text{or} \quad \sum_{a=1}^{4} N_{a}(\xi,\eta) = 1$$





## 4.4 Three-dimensional tetrahedron element

• The compatible displacement field is again given by a complete linear polynomial expansion as

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$$\phi^{e} \approx \hat{\phi}^{e} = \begin{bmatrix} 1 & x & y & z \end{bmatrix} \begin{cases} \alpha_{1} \\ \alpha_{2} \\ \alpha_{3} \\ \alpha_{4} \end{cases} = \alpha_{1} + x\alpha_{2} + y\alpha_{3} + z\alpha_{4}$$

where  $\phi^*$  is displacement on the element, which can be the displacement *u* in *x*-direction or the displacement *v* in *y*-direction or the displacement *w* in *z*-direction; the unknown parameters  $\alpha_1$  to  $\alpha_*$  may be evaluated in terms of the displacements at each of the four vertices of the tetrahedron. <sup>13</sup>



4.4 Three-dimensional tetrahedron element	
• <i>V</i> is the volume of the tetrahedron	
$6V = \det \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix}$	
$\begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix}^{-1} = \frac{1}{6V} \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$	15 15



#### 4.4 Three-dimensional tetrahedron element

• The above solution for the parameters *α<sub>a</sub>* permits the element interpolations to be rewritten in terms of nodal parameters as

$$\begin{split} \hat{\phi}^{e} &= \alpha_{1} + \alpha_{2}x + \alpha_{3}y + \alpha_{4}z \\ &= \sum_{a=1}^{4} \frac{1}{6V} \Big( a_{a}\hat{\phi}_{a} + b_{a}\hat{\phi}_{a}x + c_{a}\hat{\phi}_{a}y + d_{a}\hat{\phi}_{a}z \Big) \\ &= \sum_{a=1}^{4} \frac{1}{6V} \Big( a_{a} + b_{a}x + c_{a}y + d_{a}z \Big) \hat{\phi}_{a}^{*} \end{split}$$

• This gives the shape functions

$$N_{a}(x, y, z) = \frac{1}{6V} (a_{a} + b_{a}x + c_{a}y + d_{a}z), \quad a = 1, 2, 3, 4$$

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## 4.5 Three-dimensional hexahedron element

#### Hexahedron with eight nodes

• For a three-dimensional problem, we consider the hexahedron shown in Fig. 4.17.



#### 4.5 Three-dimensional hexahedron element

• The development of shape functions follows the procedure used for the four-node rectangle. For the derivation of the shape functions, it is convenient to use a local Cartesian system *x*', *y*', *z*' defined by

where

$$x_0 = \frac{1}{8} \sum_{a=1}^{8} x_a, \quad y_0 = \frac{1}{8} \sum_{a=1}^{8} y_a, \quad z_0 = \frac{1}{8} \sum_{a=1}^{8} z_a$$

 $x' = x - x_0, \quad y' = y - y_0, \quad z' = z - z_0$ 

in which  $x_0$ ,  $y_0$ ,  $z_0$  are located at the center of the hexahedron and  $x_a$ ,  $y_a$ ,  $z_a$  are coordinates of the nodes.

4.5 Three-dimensional hexahedron element • We may write a polynomial expression for  $\phi^e$  as  $\phi^e \approx \hat{\phi}^e = \begin{bmatrix} 1 & x' & y' & z' & x'y' & y'z' & z'x' & x'y'z' \end{bmatrix} \begin{cases} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_8 \\ \alpha_7 \\ \alpha_8 \\ \alpha_7 \\ \alpha_8 \\ \alpha_7 \\ \alpha_8 \\ \alpha_8$ 







## 4.5 Three-dimensional hexahedron element

• The coefficients α<sub>a</sub> may be obtained in the identical manner used above for the two-dimensional rectangle. Basing on the inverse to the coefficient matrix and substitution, we can obtain the eight shape functions

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$$\begin{split} N_{1} &= \frac{1}{8} (1-\xi) (1-\eta) (1-\zeta), \quad N_{5} &= \frac{1}{8} (1-\xi) (1-\eta) (1+\zeta) \\ N_{2} &= \frac{1}{8} (1+\xi) (1-\eta) (1-\zeta), \quad N_{6} &= \frac{1}{8} (1+\xi) (1-\eta) (1+\zeta) \\ N_{3} &= \frac{1}{8} (1+\xi) (1+\eta) (1-\zeta), \quad N_{7} &= \frac{1}{8} (1+\xi) (1+\eta) (1+\zeta) \\ N_{4} &= \frac{1}{8} (1-\xi) (1+\eta) (1-\zeta), \quad N_{8} &= \frac{1}{8} (1-\xi) (1+\eta) (1+\zeta) \end{split}$$

