Computational Mechanics

(*English Course*)

Yongliang Wang

School of Mechanics and Civil Engineering

China University of Mining and Technology (Beijing)

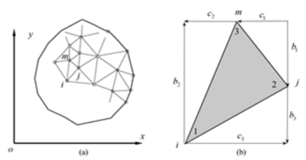
Friday, June 2, 2023

4 Elements and Shape Functions

4.2 Two-dimensional triangle element

- **Triangle with three nodes**

- The finite element domain is defined by dividing the domain into a mesh of two-dimensional triangular elements as shown in Fig. 4.4a. A simple set of linear functions can be constructed from linear polynomials over three-node triangles as shown in Fig. 4.4b.



- **Figure 4.4 Division of a two-dimensional domain into triangle elements: (a) triangle mesh, (b) triangle element.**

4.2 Two-dimensional triangle element

- The approximation in each triangle may be written as a linear function of the Cartesian coordinates:

$$\phi^e \approx \hat{\phi}^e = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \alpha_1 + \alpha_2 x + \alpha_3 y$$

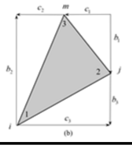
where ϕ^e is displacement on the element, which can be the displacement u in x-direction or the displacement v in y-direction; the unknown parameters α_1 to α_3 may be evaluated in terms of the displacements at each of the three vertices of the triangle.

4.2 Two-dimensional triangle element

- The vertices define the nodes of the triangle. Accordingly, we write the set of equations

$$\begin{Bmatrix} \hat{\phi}_1^e \\ \hat{\phi}_2^e \\ \hat{\phi}_3^e \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

where x_a and y_a are coordinates at the three vertices of the triangle.



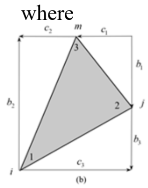
$$\phi^e \approx \hat{\phi}^e = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \alpha_1 + \alpha_2 x + \alpha_3 y$$

4.2 Two-dimensional triangle element

- The inverse to the coefficient matrix is given by

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} = \frac{1}{2\Delta} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

where



$$a_1 = x_2 y_3 - x_3 y_2, \quad b_1 = y_2 - y_3, \quad c_1 = x_3 - x_2$$
$$a_2 = x_3 y_1 - x_1 y_3, \quad b_2 = y_3 - y_1, \quad c_2 = x_1 - x_3$$
$$a_3 = x_1 y_2 - x_2 y_1, \quad b_3 = y_1 - y_2, \quad c_3 = x_2 - x_1$$

area of the triangle $\Delta = (x_1 b_1 + x_2 b_2 + x_3 b_3)/2$

4.2 Two-dimensional triangle element

- By substitution, we can get

$$\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix}=\begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{bmatrix}^{-1}\begin{Bmatrix}\hat{\phi}_1^e\\ \hat{\phi}_2^e\\ \hat{\phi}_3^e\end{Bmatrix}=\frac{1}{2\Delta}\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\begin{Bmatrix}\hat{\phi}_1^e\\ \hat{\phi}_2^e\\ \hat{\phi}_3^e\end{Bmatrix}=\frac{1}{2\Delta}\begin{Bmatrix}\sum_{a=1}^{3}a_a\hat{\phi}_a^e\\ \sum_{a=1}^{3}b_a\hat{\phi}_a^e\\ \sum_{a=1}^{3}c_a\hat{\phi}_a^e\end{Bmatrix}$$

- The above solution for the parameters α_a permits the element interpolations to be rewritten in terms of nodal parameters as

$$\hat{\phi}^e=\alpha_1+\alpha_2x+\alpha_3y=\sum_{a=1}^{3}\frac{1}{2\Delta}\left(a_a\hat{\phi}_a+b_a\hat{\phi}_ax+c_a\hat{\phi}_ay\right)=\sum_{a=1}^{3}\frac{1}{2\Delta}\left(a_a+b_ax+c_ay\right)\hat{\phi}_a^e$$

$$\phi^e\approx\hat{\phi}^e=\begin{bmatrix}1 & x & y\end{bmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix}=\alpha_1+\alpha_2x+\alpha_3y$$

4.2 Two-dimensional triangle element

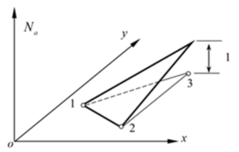
- Thus, the three shape functions for the triangle are given by

$$N_a(x,y)=\frac{1}{2\Delta}\left(a_a+b_ax+c_ay\right),\quad a=1,2,3$$

$$\hat{\phi}^e=\alpha_1+\alpha_2x+\alpha_3y=\sum_{a=1}^{3}\frac{1}{2\Delta}\left(a_a\hat{\phi}_a+b_a\hat{\phi}_ax+c_a\hat{\phi}_ay\right)=\sum_{a=1}^{3}\frac{1}{2\Delta}\left(a_a+b_ax+c_ay\right)\hat{\phi}_a^e$$

4.2 Two-dimensional triangle element

- The shape function for a=3 is shown in Fig. 4.5a., it can be seen that the value of this shape function is one at node 3, and the value is zero at other nodes 1 or 2. More generally, these shape functions have the following properties



$$N_a\left(x_b,y_b\right)=\delta_{ab}=\begin{cases}1, & a=b\\ 0, & a\neq b\end{cases}$$

where $\left(x_b,y_b\right)$ is local coordinate.

- **Figure 4.5 Shape function N_3 for two-dimensional triangle with three nodes.**

4.2 Two-dimensional triangle element

- Besides, the completeness condition then requires that $\hat{\phi}^e(x_b,y_b)$ contains any constant c (displacement of rigid body), which then yields

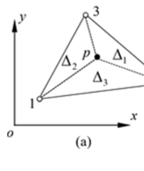
$$\hat{\phi}^e(x,y)=\sum_{a=1}^{3}N_a(x,y)\,c=c\quad\text{or}\quad\sum_{a=1}^{3}N_a(x,y)=1$$

- Using these shape functions, we can write the set of approximations in each individual element as

$$\hat{\phi}^e=\sum_{a=1}^{3}N_a(x,y)\hat{\phi}_a^e$$

4.2 Two-dimensional triangle element

- As an example let us consider again the two-dimensional plane problem for which a simple set of linear functions for a parameter ϕ may be constructed from linear polynomials over three-node triangles as shown in Fig. 4.6a.



- **Figure 4.6 Triangular element with area coordinates: (a) subareas of triangle, (b) area coordinates..**

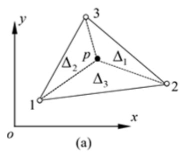
4.2 Two-dimensional triangle element

- Geometrically, the numerator of the shape function is twice the area for the triangle composed of point (x, y) and two of the nodal coordinates x_a, y_a.

$$2\Delta_a(x,y)=\left(a_a+b_ax+c_ay\right)$$

$$\Delta=\left(x_1b_1+x_2b_2+x_3b_3\right)/2$$

$$N_a(x,y)=\frac{1}{2\Delta}\left(a_a+b_ax+c_ay\right),\quad a=1,2,3$$



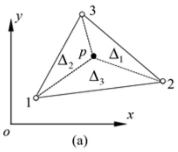
- **Figure 4.6 Triangular element with area coordinates: (a) subareas of triangle, (b) area coordinates..**

4.2 Two-dimensional triangle element

- The shape functions become

$$N_a = \frac{\Delta_a(x, y)}{\Delta}, \quad a = 1,2,3$$

This ratio also defines a convenient coordinate system (Fig. 4.6b) for triangles which is known as area coordinates.



- **Figure 4.6 Triangular element with area coordinates: (a) subareas of triangle, (b) area coordinates..**

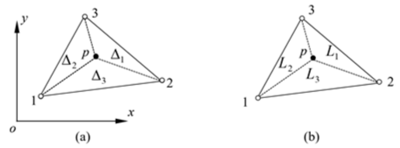
4.2 Two-dimensional triangle element

- We shall denote these as

$$L_a = \frac{\Delta_a}{\Delta}, \quad a = 1,2,3$$

- Since there are three coordinates it is necessary to always note that they are constrained by the total area such that

$$L_1 + L_2 + L_3 = 1$$



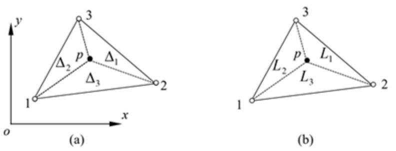
- **Figure 4.6 Triangular element with area coordinates: (a) subareas of triangle, (b) area coordinates..**

4.2 Two-dimensional triangle element

- Using area coordinates, the shape functions for the three-node triangle are given by

$$N_a = L_a, \quad a = 1,2,3$$

$$L_a = \frac{\Delta_a}{\Delta}, \quad a = 1,2,3 \qquad \Delta_a(x, y) = (a_a + b_a x + c_a y)/2$$

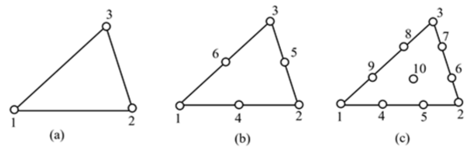


- **Figure 4.6 Triangular element with area coordinates: (a) subareas of triangle, (b) area coordinates..**

4.2 Two-dimensional triangle element

- **Higher order triangle element**

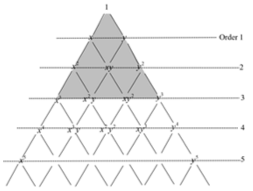
- The process originally used for the three-node triangle may be performed for triangular elements with more than three nodes. The first three members of the triangular element family are shown in Fig. 4.7.



- **Figure 4.7 Triangle element family: (a) linear, (b) quadratic, (c) cubic.**

4.2 Two-dimensional triangle element

- It is useful to note that a complete set of polynomials in two dimensions may be represented on a Pascal triangle as shown in Fig. 4.8. Using a Pascal triangle, we note that each complete order of the polynomials exactly matches the number of nodes of the triangle with the same order.



- **Figure 4.8 Triangular element with area coordinates: (a) subareas of triangle, (b) area coordinates..**

4.2 Two-dimensional triangle element

- Thus, the shape functions for a six-node triangle may be obtained using quadratic order polynomials as

$$\phi^e \approx \hat{\phi}^e = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix}$$

$$= \alpha_1 + x\alpha_2 + y\alpha_3 + x^2\alpha_4 + xy\alpha_5 + y^2\alpha_6$$

4.2 Two-dimensional triangle element

- Pursuing this approach will require the inverse of a 6×6 matrix to obtain the expression for the parameters α_a in terms of the nodal parameters $\hat{\phi}_a^e$. We shall find it useful to develop the shape functions directly using area coordinates and, thus, eliminate the need to invert matrices.

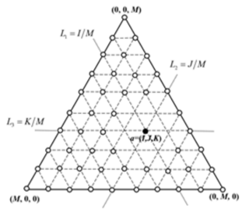
$$\phi^e \approx \hat{\phi}^e = \alpha_1 + x\alpha_2 + y\alpha_3 + x^2\alpha_4 + xy\alpha_5 + y^2\alpha_6$$

Triangle with three nodes

$$\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix} = \begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{bmatrix}^{-1}\begin{Bmatrix}\hat{\phi}_1^e\\ \hat{\phi}_2^e\\ \hat{\phi}_3^e\end{Bmatrix} = \frac{1}{2\Delta}\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\begin{Bmatrix}\hat{\phi}_1^e\\ \hat{\phi}_2^e\\ \hat{\phi}_3^e\end{Bmatrix} = \frac{1}{2\Delta}\begin{Bmatrix}\sum_{a=1}^{3} a_a\hat{\phi}_a^e\\ \sum_{a=1}^{3} b_a\hat{\phi}_a^e\\ \sum_{a=1}^{3} c_a\hat{\phi}_a^e\end{Bmatrix}$$

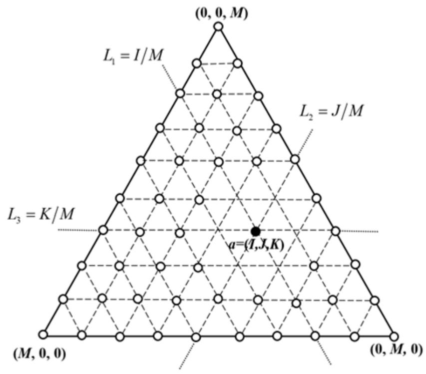
4.2 Two-dimensional triangle element

- To derive shape functions for higher order elements, it is simple to write an arbitrary triangle of order M in a direct manner. As shown in Fig. 4.9, denoting a typical node a by three numbers I, J, and K corresponding to the position of coordinates L_{1a}, L_{2a}, and L_{3a}



- **Figure 4.9 A general triangular element.**

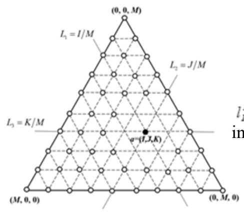
4.2 Two-dimensional triangle element



- **Figure 4.9 A general triangular element.**

4.2 Two-dimensional triangle element

- To derive shape functions for higher order elements, it is simple to write an arbitrary triangle of order M in a direct manner. As shown in Fig. 4.9, denoting a typical node a by three numbers I, J, and K corresponding to the position of coordinates L_{1a}, L_{2a}, and L_{3a}, we can write the shape function in terms of three Lagrangian interpolations as



$$N_a = l_I^I(L_1)\, l_J^J(L_2)\, l_K^K(L_3)$$
with $I + J + K = M$

l_I^I, l_J^J, l_K^K are given by Lagrangian interpolations, L_1, L_2, L_3 taking the place of ξ.

- **Figure 4.9 A general triangular element.**

4.2 Two-dimensional triangle element

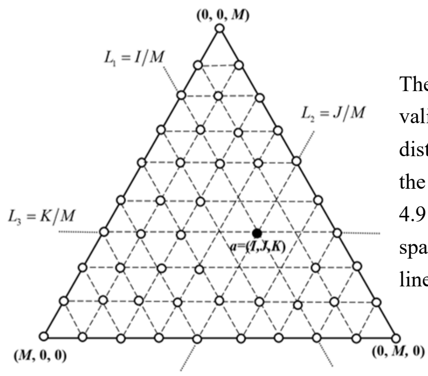
- The shape function in terms of three Lagrangian interpolations

$$N_a = l_I^I(L_1)\, l_J^J(L_2)\, l_K^K(L_3)$$
with $I + J + K = M$

l_I^I, l_J^J, l_K^K are given by Lagrangian interpolations, L_1, L_2, L_3 taking the place of ξ.

$$l_a^p(\xi) = \prod_{\substack{b=1\\ b\neq a}}^{n} \frac{(\xi - \xi_b)}{(\xi_a - \xi_b)} = \frac{(\xi-\xi_1)(\xi-\xi_2)(\cdots)(\xi-\xi_{a-1})(\xi-\xi_{a+1})(\cdots)(\xi-\xi_n)}{(\xi_a-\xi_1)(\xi_a-\xi_2)(\cdots)(\xi_a-\xi_{a-1})(\xi_a-\xi_{a+1})(\cdots)(\xi_a-\xi_n)}$$

4.2 Two-dimensional triangle element



The above equaiton is valid for quite arbitrary distributions of nodes of the pattern given in Fig. 4.9 and simplifies if the spacing of the nodal lines is equal (i.e., $1/M$) .

- **Figure 4.9 A general triangular element.**

4.2 Two-dimensional triangle element

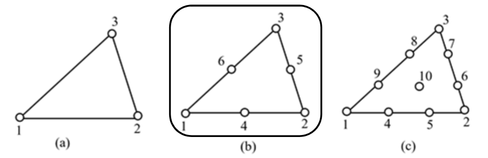
- Quadratic triangle element

  Corner nodes:

  $$N_a = (2L_a - 1)L_a, \quad a = 1, 2, 3$$

  Mid-side nodes:

  $$N_4 = 4L_1L_2, \quad N_5 = 4L_2L_3, \quad N_6 = 4L_3L_1$$



- **Figure 4.7 Triangle element family: (a) linear, (b) quadratic, (c) cubic.**

46

4.2 Two-dimensional triangle element

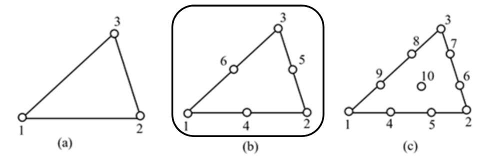
- Quadratic triangle element

  Node 4

  I=1, J=1, K=0

  M=I+J+K=2

$$\begin{aligned} N_4 &= l_I^I(L_1)\, l_J^J(L_2)\, l_K^K(L_3) \\ &= l_1^1(L_1)\, l_1^1(L_2)\, l_0^0(L_3) = l_1^1(L_1) \times l_1^1(L_2) \times 1 \\ &= \frac{L_1 - L_1(\xi_2)}{L_1(\xi_4) - L_1(\xi_2)} \frac{L_2 - L_2(\xi_1)}{L_2(\xi_4) - L_2(\xi_1)} \\ &= \frac{L_1 - 0}{1/2 - 0} \frac{L_2 - 0}{1/2 - 0} = 4L_1L_2 \end{aligned}$$



- **Figure 4.7 Triangle element family: (a) linear, (b) quadratic, (c) cubic.**

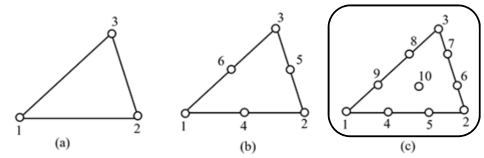
47

4.2 Two-dimensional triangle element

- Cubic triangle element

  Corner nodes:

  $$N_a = \frac{1}{2}(3L_a - 1)(3L_a - 2)L_a, \quad a = 1, 2, 3$$



- **Figure 4.7 Triangle element family: (a) linear, (b) quadratic, (c) cubic.**

48

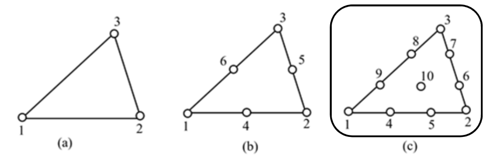
4.2 Two-dimensional triangle element

- Cubic triangle element

  Mid-side nodes:

  $$N_4 = \frac{9}{2}L_1L_2(3L_1 - 1), \quad N_5 = \frac{9}{2}L_1L_2(3L_2 - 1), \quad N_6 = \frac{9}{2}L_2L_3(3L_2 - 1)$$

  $$N_7 = \frac{9}{2}L_2L_3(3L_3 - 1), \quad N_8 = \frac{9}{2}L_3L_1(3L_3 - 1), \quad N_9 = \frac{9}{2}L_3L_1(3L_1 - 1)$$



- **Figure 4.7 Triangle element family: (a) linear, (b) quadratic, (c) cubic.**

49

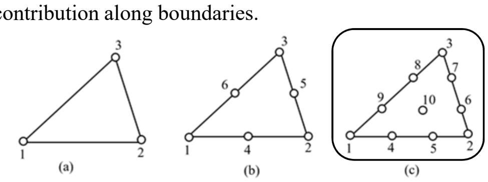
4.2 Two-dimensional triangle element

- Cubic triangle element

  The internal node:

  $$N_{10} = 27L_1L_2L_3$$

  The last shape again is a “bubble” function giving zero contribution along boundaries.



- **Figure 4.7 Triangle element family: (a) linear, (b) quadratic, (c) cubic.**

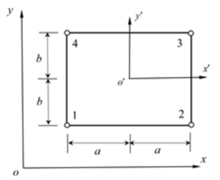
50

4 Elements and Shape Functions

4.3 Two-dimensional rectangle element

■ **Linear rectangle element with four nodes**

- As a second example of two-dimensional shape functions, we consider rectangles of the form shown in Fig. 4.10. The rectangular element considered has side lengths of 2*a* and 2*b* in the *x*- and *y*-directions, respectively.

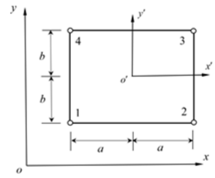


- **Figure 4.10 Rectangle element geometry and node numbers.**

4.3 Two-dimensional rectangle element

- For the derivation of the shape functions it is convenient to use a local Cartesian system *x'*, *y'* defined by

$$x' = x - x_0 \quad \text{and} \quad y' = y - y_0$$



$$x_0 = \frac{1}{4}\sum_{\alpha=1}^{4} x_a \quad \text{and} \quad y_0 = \frac{1}{4}\sum_{\alpha=1}^{4} y_a$$

in which x_0, y_0 are located at the center of the rectangle and x_a, y_a are coordinates of the nodes.

- **Figure 4.10 Rectangle element geometry and node numbers.**

4.3 Two-dimensional rectangle element

- We now need four functions for each displacement component in order to uniquely define the shape functions. A suitable choice is given by:

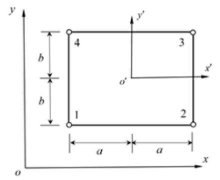
$$\phi^e \approx \hat{\phi}^e = [1 \quad x' \quad y' \quad x'y'] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \alpha_1 + x'\alpha_2 + y'\alpha_3 + x'y'\alpha_4$$

where ϕ^e is displacement on the element, which can be the displacement *u* in *x*-direction or the displacement *v* in *y*-direction; the unknown parameters α_1 to α_4 may be evaluated in terms of the displacements at each of the four vertices of the rectangle.

4.3 Two-dimensional rectangle element

- The coefficients α_a may be obtained at each vertex node giving

$$\begin{Bmatrix} \hat{\phi}_1^e \\ \hat{\phi}_2^e \\ \hat{\phi}_3^e \\ \hat{\phi}_4^e \end{Bmatrix} = \begin{bmatrix} 1 & -a & -b & ab \\ 1 & a & -b & -ab \\ 1 & a & b & ab \\ 1 & -a & b & -ab \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$



- **Figure 4.10 Rectangle element geometry and node numbers.**

4.3 Two-dimensional rectangle element

- Basing on the inverse to the coefficient matrix and substitution, we can again solve for α_a in terms of the nodal displacements to obtain finally

$$\hat{\phi}^e = \frac{1}{4}\left(1-\frac{x'}{a}\right)\left(1-\frac{y'}{b}\right)\hat{\phi}_1^e + \frac{1}{4}\left(1+\frac{x'}{a}\right)\left(1+\frac{y'}{b}\right)\hat{\phi}_2^e + \frac{1}{4}\left(1+\frac{x'}{a}\right)\left(1+\frac{y'}{b}\right)\hat{\phi}_3^e + \frac{1}{4}\left(1-\frac{x'}{a}\right)\left(1-\frac{y'}{b}\right)\hat{\phi}_4^e$$

- We obtain the four shape functions

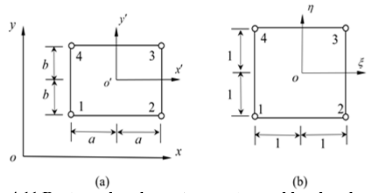
$$N_1 = \frac{1}{4}\left(1-\frac{x'}{a}\right)\left(1-\frac{y'}{b}\right), \quad N_2 = \frac{1}{4}\left(1+\frac{x'}{a}\right)\left(1-\frac{y'}{b}\right)$$

$$N_3 = \frac{1}{4}\left(1+\frac{x'}{a}\right)\left(1+\frac{y'}{b}\right), \quad N_4 = \frac{1}{4}\left(1-\frac{x'}{a}\right)\left(1+\frac{y'}{b}\right)$$

4.3 Two-dimensional rectangle element

- For implementing the numerical integration to evaluate the integrals in weak form and providing the rectangular element geometry and local node numbers as shown in Fig. 4.11, we let

$$\xi = \frac{x'}{a} \text{ and } \eta = \frac{y'}{b} \qquad -1 \le \xi, \eta \le 1$$



- **Figure 4.11 Rectangular element geometry and local node numbers: (a) global coordinates, (b) local coordinates.**

4.3 Two-dimensional rectangle element

- The shape functions may be written as

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta),\quad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta),\quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$N_1 = \frac{1}{4}\left(1-\frac{x'}{a}\right)\left(1-\frac{y'}{b}\right),\quad N_2 = \frac{1}{4}\left(1+\frac{x'}{a}\right)\left(1-\frac{y'}{b}\right)$$

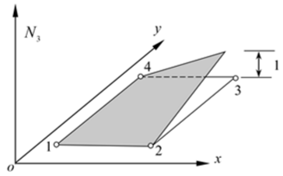
$$N_3 = \frac{1}{4}\left(1+\frac{x'}{a}\right)\left(1+\frac{y'}{b}\right),\quad N_4 = \frac{1}{4}\left(1-\frac{x'}{a}\right)\left(1+\frac{y'}{b}\right)$$

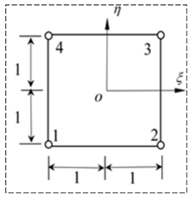
$$\xi = \frac{x'}{a} \text{ and } \eta = \frac{y'}{b} \qquad -1 \le \xi, \eta \le 1$$

4.3 Two-dimensional rectangle element

- The shape function for a=3 is shown in Fig. 4.12, it can be seen that the value of this shape function is one at node 3, and the value is zero at other nodes 1, 2, or 4.

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$





- **Figure 4.12 Shape function N_3 for two-dimensional rectangle with four nodes.**

4.3 Two-dimensional rectangle element

- More generally, these shape functions have the following properties:

$$N_a(\xi_b, \eta_b) = \delta_{ab} = \begin{cases} 1, & a = b \\ 0, & a \ne b \end{cases}$$

where (ξ_b, η_b) is the local coordinate.

- Besides, the completeness condition then requires that $\hat{\phi}^e(\xi,\eta)$ contains any constant c (displacement of rigid body), which then yields

$$\hat{\phi}^e(\xi,\eta) = \sum_{a=1}^{4} N_a(\xi,\eta)\, c = c \quad \text{or} \quad \sum_{a=1}^{4} N_a(\xi,\eta) = 1$$

4.3 Two-dimensional rectangle element

■ **Higher order rectangle element**

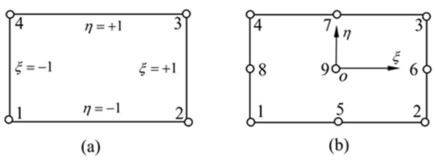
- Based on the above a systematic and easy method to generate shape functions for any order of rectangle can be achieved by a simple product of Lagrange interpolations in the two local coordinates

$$N_a \equiv N_{IJ} = l_I^n(\xi)\, l_J^m(\eta)$$

4.3 Two-dimensional rectangle element

$$N_a \equiv N_{IJ} = l_I^n(\xi)\, l_J^m(\eta)$$

where n and m stand for the number of subdivisions in each direction.



- **Figure 4.13 Rectangle element family: (a) linear, (b) quadratic.**

4.3 Two-dimensional rectangle element

$$N_a \equiv N_{IJ} = l_I^n(\xi)\, l_J^m(\eta)$$

Table 4.1 shows how the I, J values are mapped to the node numbers a for rectangle with four nodes.

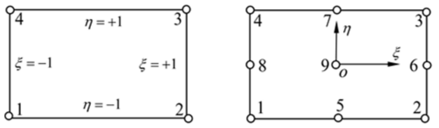
- **Table 4.1 Numbering for rectangle with four nodes.**

Label	Node number			
a	1	2	3	4
I	1	2	2	1
J	1	1	2	2

4.3 Two-dimensional rectangle element

- Figure 4.13 shows a few members of the family where m=n. For m=n =1, we obtain the simple result

$$N_a = \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta)$$



- **Figure 4.13 Rectangle element family: (a) linear, (b) quadratic.**

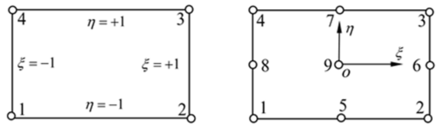
64

4.3 Two-dimensional rectangle element

■ **Quadratic rectangle element**

- For m=n =2, the quadratic rectangle element with nine nodes is shown as Fig. 4.13b.

$$N_a \equiv N_{IJ} = l_I^n(\xi)\, l_J^m(\eta)$$



- **Figure 4.11 Rectangle element family: (a) linear, (b) quadratic.**

65

4.3 Two-dimensional rectangle element

- Table 4.2 shows how the I, J values are mapped to the node numbers a for quadratic rectangle element

$$N_a \equiv N_{IJ} = l_I^n(\xi)\, l_J^m(\eta)$$

- **Table 4.2 Numbering for quadratic rectangle element.**

Label	Node number								
a	1	2	3	4	5	6	7	8	9
I	1	3	3	1	2	3	2	1	2
J	1	1	3	3	1	2	3	2	2

66

4.2 Two-dimensional rectangle element

- Quadratic rectangle element

Corner nodes

$$N_a = \frac{1}{4}\xi\eta(\xi+\xi_a)(\eta+\eta_a), \quad a = 1, 2, 3, 4$$

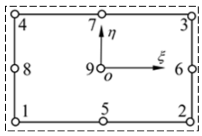
Mid-side nodes

$$\xi_a = 0,\ N_a = \frac{1}{2}\eta(1-\xi^2)(\eta+\eta_a), \quad a = 5, 7$$

$$\eta_a = 0,\ N_a = \frac{1}{2}\xi(\xi+\xi_a)(1-\eta^2), \quad a = 6, 8$$

Center node

$$N_9 = (1-\xi^2)(1-\eta^2)$$



67

The End